M/A 191.

ELEMENTARY MATHEMATICS

A. M. BOZMAN, A.E. DENMARK
AND E. M. TRICKEY

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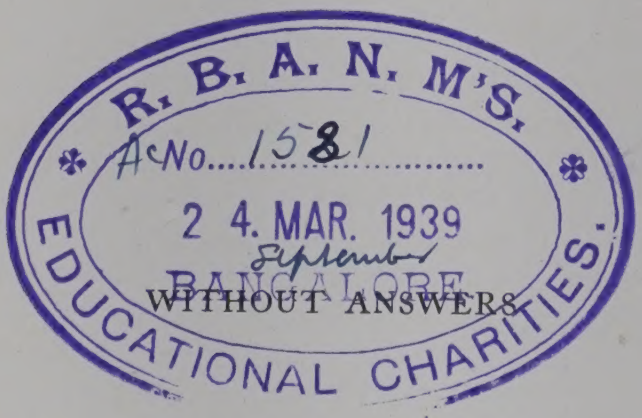
A Combined Course in Arithmetic, Algebra, and Geometry for Junior Forms

BY

A. M. BOZMAN, M.A., A. E. DENMARK, B.A.,
AND E. M. TRICKEY, B.A.

ASSISTANT MISTRESSES, THE COUNTY SECONDARY SCHOOL, STREATHAM



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PREFACE

THE aim of this text-book is to provide a combined course in Arithmetic, Algebra, and Geometry for use in the junior forms of Secondary Schools. It presents also a scheme of work which would be useful for pupils in Central Schools and the supplementary forms of Elementary Schools.

The arithmetic section is designed to cover the main work needed for the General School Examination, and consequently forms a large portion of the course. The algebra and geometry provide a sound basis on which the subsequent algebra and formal geometry for the Matriculation Examination may be developed. The formal geometry included follows a fairly comprehensive course in practical geometry in the second forms. In this section of the work the aim has been to emphasise, in each individual exercise, the necessity for a rigorously logical proof sequence based on the data of that exercise and what has already been proved. The actual exercise sequence is an arbitrary consideration, and here is based on those " assumptions " the pupils have already tested, not proved, in the second forms.

While this book does not fail to meet the requirements of ordinary class teaching, it is especially suitable for schools where individual methods and investigation on the part of the pupil are encouraged. It does indeed owe its conception to the complete lack of suitable text-books for use in the junior forms of a school run on a modified Dalton plan. With this end in view, the explanatory articles have been made detailed and as simple as is consistent with

the main purpose of a sound mathematical outlook, so that it may be possible for the pupil to grasp them during study periods.

Answers have been given in all cases, as their proper use is considered stimulating to the pupils, who are thus enabled to judge for themselves their own degree of understanding of the work in hand. The exercises have been graded, and throughout the whole course there has been a sustained effort to co-ordinate the three branches of the subject.

We should like to express our thanks to Mr Benchara Branford, Divisional Inspector of the London County Council, to whose instigation our work owes its inception, and without whose advice and encouragement in its earlier stages it would hardly have been completed, and to Mr B. A. Howard, Head Master of the Addey and Stanhope School, whose criticism and counsel have been invaluable, and who has most generously placed his time and wide experience at our disposal.

A. M. B.
A. E. D.
E. M. T.

April 1927.

As the Authors of this Text-book are in the service of the London County Council, it is proper to state that the Council is in no way responsible for the contents of the book, and has taken no part in its publication.

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Elementary Mathematics

A Combined Course in Arithmetic, Algebra and Geometry

CHAPTER I

Equations

1. Suppose someone gave me a bag of apples. I should want to know how many apples there were in the bag, and I might be told in any one of the following ways :—

(1). There are 10 apples in the bag.

(2). If you had 3 such bags of apples, you would have 30 apples.

(3). If you had 2 more apples in the bag, you would have a dozen apples.

(4). If the number of apples in the bag had been 6 less, you would have had 4 apples.

Now these would all have been different ways of telling me the same fact. The first way is the easiest, but the others would tell me the number quite correctly if I took the trouble to use the facts that I had been told.

2. Sometimes we have to find out a number which we cannot be told straight away. Then, to obtain the number, we must be able to use the facts we have been told about it.

We first write a sign for it. The signs most generally used are the letters of the alphabet, and the one most often used is x. If we were going to use x for the number of apples spoken of in paragraph 1, we should begin :—

Let x represent the number of apples in the bag.

Then we could write :—

(1). $x=10$.

You see that we have used the sign $=$.

We call this sign **equals**.

When we use this method of saying that two numbers or two quantities are equal, we say that we have written an **equation**.

We shall now write the facts that are given in (2), (3), and (4) as equations.

(2). 3 times $x=30$.

Now it is clear that 3 times x is not a very convenient way of writing this number, so we write $3x$ instead. This means 3 multiplied by x.

Our simple way of writing equation (2) is, therefore, $3x=30$.

Equation (3) is a simple one, $x+2=12$.

Equation (4) is rather like it, $x-6=4$.

3. We might have been given any one of these equations and, from it, could have found the answer to our problem.

(1) tells us the answer at once, x is 10.

From (2) we see that, if $3x=30$, x must be 10.

From (3) we see that the number which, increased by 2, gives 12, is 10.

From (4) we see that the number which becomes 4 when 6 is taken away, is 10.

4. When we find out the value of the unknown number from an equation, we say that we have **solved** the equation.

Now let us look at one of our equations and at our solution of it.

$$3x=30,$$
$$\therefore\ x=10 \quad . \quad . \quad . \quad . \quad (1)$$

If $x=4$, would $3x=30$? No.

If $x=25$, would $3x=30$? No.

In short, $3x$ equals 30 only when $x=10$.

That is, the value of x in (1) gives us the only condition under which $3x=30$.

When we solve an equation, we are finding the condition which will make the equation true.

We can easily check the answers to our equations, as, if our condition is the correct one, then the right-hand side must equal the left-hand side if we put in our value for x.

Thus :—The left-hand side is $3x$,

which $=30$ (when $x=10$).

The right-hand side is 30.

That is, the left-hand side is equal to the right-hand side on the condition that $x=10$.

Examples Ia (*oral*)

What must x equal if :—

1. $4x=36$.	6. $3x+5=26$.	11. $3x+2=17$.
2. $4x+3=39$.	7. $3x-5=16$.	12. $3x-5=10$.
3. $4x-3=33$.	8. $x+5=11$.	13. $x+7=12$.
4. $3x+x+6=42$.	9. $3x=15$.	14. $x-5=9$.
5. $3x=21$.	10. $2x+x=15$.	15. $5x=20$.

5. We have not found it difficult to find the conditions under which these equations (Examples Ia) are true. We have not been told the value of x in any one of them, but what we have been told about this value has made it easy for us to find it. Let us examine some more of these equations.

EXAMPLE 1.—*Solve* $x+5=49$ . . . . . (1)

That is, find the condition which makes this equation true.

In this case we have to find the number which, when added to 5, will give 49.

What is this number ?

If this condition is written in equation form, we have $x=44$.

How does this compare with the original equation ?

We see that we have taken 5 from each side.

We could write this out fully as

$$x+5-5=49-5 \quad . \quad . \quad . \quad . \quad (2)$$

or $$x=44.$$

In working out this simple equation (1) it is obvious that line (2) can be omitted, but it introduces an important rule about equations, namely,

The condition which makes an equation true is unaltered, if we add the same quantity to both sides, or if we subtract the same quantity from both sides.

EXAMPLE 2.—*Let us look at the equation* $4x-3=33$.

The left-hand side of this equation is 3 less than $4x$. If we alter this side to $4x$, how have we changed it ? In order that the equation may still be true, how must we change the right-hand side ?

What is the value of x which satisfies the equation ?

EXAMPLE 3.—*Solve* $5x-7=43$.

Then $$5x= \ ?$$

$$\therefore \ x= \ ?$$

EXAMPLE 4.—*Solve* $6x-9+3=30$.

Then $6x = ?$

$\therefore\ x = ?$

Examples Ib

Find the values of x which satisfy the following equations, and check all answers.

Lower.

1. $3x+5=26$.
2. $2x-11=37$.
3. $2x+7+5=36$.
4. $9x-8-6=31$.
5. $6x-5+11=42$.
6. $2+3x-7+8=54$.
7. $7+8x-12=59$.
8. $3+5x-11=92$.
9. $7x+16-4+2=28$.
10. $11x+8-12+6-5=85$.
11. $9x+7-21+36-61=0$.
12. $19+3x-4+7=34$.
13. $21+6-40+9+7x=3$.
14. $5+4x+10+2=29$.
15. $7x+3-10+32=60$.
16. $11x-17+4-20=0$.
17. $14x+6+10-23=7$.
18. $6-9+15+3x=24$.
19. $13x+11-7-30=0$.
20. $10x+20+7-19=82$.
21. $5-25+80+4x=80$.
22. $12x-18+57-12=92$.
23. $14+7x-22+18=63$.
24. $19-5+3x-51=82-12$.
25. $4x-52+17-63=0$.
26. $33-58+90+10x=82$.
27. $3x+12-77+54=90$.
28. $72-45+5x=65+24$.
29. $8x-22-57+3=82$.
30. $3x-34+72-81=26$.

Middle.

31. $12x+19-6+12=25$.
32. $62x-43+117=198$.
33. $10x+21-76+45=0$.

Higher.

34. $71-143+6x+72=0$.
35. $51+16x+32-11=88$.
36. $3x+6\frac{1}{2}-4\frac{1}{2}+1\frac{1}{4}=7$.
37. $4x+4{\cdot}52-8{\cdot}73=2{\cdot}13$.
38. $8\frac{2}{3}+5x-4\frac{1}{3}=5\frac{2}{3}$.
39. $16{\cdot}8-12{\cdot}42+5x=13{\cdot}7$.
40. $8+2x-6\frac{1}{7}+4\frac{2}{7}-9=21\frac{6}{7}$.

6. We deal with equations which have more than one x-term in the same way.

EXAMPLE 1.—*Solve* $5x+3x+2=26$.

This gives $8x=26-2$;

i.e. $8x=24$,

and $x=3$.

Check.—The left-hand side is $5x+3x+2$,

which $=15+9+2$ (when $x=3$)
$=26$
$=$the right-hand side.

That is, the left-hand side is equal to the right-hand side on the condition that $x=3$.

EXAMPLE 2.—*Solve* $4x+12-6x=20-4x$.

This gives $4x-6x+4x=20-12$.
i.e. $2x=8$,
and $x=4$.

Check.—The left-hand side is $4x+12-6x$,

which $=16+12-24$ (when $x=4$)
$=4$.

The right-hand side is $20-4x$,

which $=20-16$ (when $x=4$)
$=4$.

That is, the left-hand side is equal to the right-hand side on the condition that $x=4$.

Examples Ic

Find the values of x which satisfy the following equations, and check all answers.

Lower.

1. $4x+7x+9=31$.
2. $8x-6x-12=64$.
3. $19-5x+11x-4=51$.
4. $x-12=6-5x$.
5. $7x-11+5x+6=4x+3$.
6. $3x+7x-4x+9-15=84$.
7. $21-5x+12+14x=8x+52$.
8. $5x+10-3x-7=4x+18-7x$.
9. $12+x-4x=3x+9-7x+15$.
10. $5x+9-3x+11-6=19-4x+31$.

Middle.

11. $43-7x+12-16+10x-4x=42-2x$.
12. $16-5x+12x-26-2x=0$.
13. $21+3x-5x+8x-69=0$.
14. $19x+11-15x-7=2x-8x-16+30$.

Higher.

15. $39x-65-2x+21=21x-15+8x+3.$
16. $77+11x-16x+12-42x-15=81-3x-2+5x-50x.$
17. $63x+81-15x+19-25=42-7x+15x.$
18. $42x+6\frac{1}{2}-82-12\frac{1}{2}+17x+81x=216x-12+14x.$
19. $8x-5{\cdot}345+18x-12x+10{\cdot}625=4x-22x+18{\cdot}6.$
20. $21x-17{\cdot}2+28x-12x+21{\cdot}57=14x+33{\cdot}7-5x.$

7. Problems.

EXAMPLE 1.—*Mary has* 6 *more marks than Joan and* 4 *less than Kathleen. The sum of their marks is* 64. *How many did each obtain?*

In this case we let x represent the number of Mary's marks.

Then $x-6$ represents the number of Joan's marks,
and $x+4$,, ,, ,, Kathleen's marks.

Now the sum of these is 64.

That is $x+x-6+x+4=64.$

$\therefore\ 3x=64-4+6;$

i.e. $3x=66,$

and $x=22.$

Although we have solved our equation, we have not answered our problem.

Since x or 22 gives the number of Mary's marks,
then Joan must have $x-6$ or 16,
and Kathleen ,, ,, $x+4$ or 26.

Our complete answer is therefore :—

Joan has 16 marks.
Mary ,, 22 ,,
Kathleen ,, 26 ,,

Check.—Adding these three results we see that the total is 64 as required.

Examples Id

Lower.

1. Mary and Jeanne together scored 19 in a game of rounders. Mary scored 5 more than Jeanne. How many did each score?

2. The Gymnastic Competition result for the second forms was as follows :—

2 A and 2 D had the same number of marks.
2 B had 10 more marks than 2A, and
2 C had 15 less marks than 2 A.

How many marks did each form obtain if the total number of marks gained by the four forms was 307 ?

3. John had 17 more marbles than Tom, Tom bought twice as many as he had before, and then had 3 more marbles than John. How many did Tom have at first ?

4. Dorothy is 4 years younger than Joyce, and Joyce is 8 years older than Mabel. The sum of their ages is 36 years. How old are they ?

5. Joan, Hilda, and Doris run a race of 100 yd. Joan wins the race, and the sum of the distances run by the three girls when Joan reaches the winning-post is 257 yd. Doris is then 3 yd. ahead of Hilda. How far had Hilda and Doris each run ?

Middle.

6. Marie bought 6 yd. of blue material and 4 yd. of pink material. She spent thirty-four shillings altogether. If the pink material cost one shilling per yd. more than the blue material, what was the price per yd. of each material ?

7. A schoolgirl buys 10 pencils, 2 pens, and an india-rubber. Each pencil costs the same as a pen, but the india-rubber costs 6d. more than a pencil. If the total cost is 2s. 8d., how much does the india-rubber cost ?

8. The school results at the end of one term are given by the following data :—

Orange House	200	marks	more than	Light Blue House.
Grey House	450	,,	,,	,, ,,
Buff House	100	,,	less ,,	,, ,,
Green House	100	,,	,, ,,	Grey House.
Dark Blue House	300	,,	more ,,	Buff House.

The total number of marks gained by all the Houses was 13,100. How many did each House obtain ?

Higher.

9. The total amount collected for the Guild by the third, fourth, and fifth forms was £60. The fourth forms together collected £7 more than the third forms, and the fifth forms together collected £10 more than the fourth forms. How much did the fourth forms collect ? If there were 3 fifth forms, and 2 of these collected equal amounts while the other collected £1 less, how much did each fifth form collect ?

CHAPTER II

Equations—(*Continued*)

8. If you were asked to draw a diagram to represent the cycle rides which two girls take one Saturday afternoon, you would find that there were several facts you would have to be told:

(1). The points from which they started.

(2). The directions in which they went.

(3). The distance, or time and rate at which they cycled.

The point we want to consider at the moment is the importance of the direction.

If any number of miles east is represented in a diagram by a line 1 in. long drawn from the starting-point towards the right edge of the paper, then the same number of miles west would be represented by a line 1 in. long drawn from the starting-point towards the left edge of the paper.

In short, lines of equal length drawn in opposite directions on the paper would represent the same distance in opposite directions of travel.

9. Sometimes we wish to show that two numbers represent measurements in opposite directions, and yet we do not want to write this out in full, nor yet to draw a diagram.

We use the following method to express this:—

Suppose $+5$ cm. means 5 cm. in one direction, then -5 cm. will mean 5 cm. in the opposite direction.

If $+5°$ means an increase of $5°$ in temperature, then $-5°$ would mean the opposite, that is, a fall of $5°$ in temperature.

If $+10$ years means 10 years from now, then -10 years mean 10 years ago.

If $+16$ miles means 16 miles north, then -16 miles means 16 miles south—and so on.

10. These two signs, $+$, $-$, are called respectively the **positive** and **negative** signs, and they mean the opposite of one another.

Whatever direction is represented by the + sign, the opposite direction is represented by the − sign.

It is usual to omit the + sign unless the meaning would not be clear without it. Otherwise it is understood. For example, 16 means +16. The negative sign must always be written.

11. The signs + and − not only tell us the directions in which measurements have been taken, they are used also to tell us what *to do* with numbers, and then they mean exactly opposite commands. The sign + between two numbers means that they are to be added, the sign − means they are to be subtracted.

$x+a+b$ means that to x we are to add a and add b.

$x-a-b$ means that from x we are to subtract first a and then b.

This is quite easy to understand until we come to the slightly more difficult matter of using negative numbers negatively.

Let us see what this means.

(1). $10+4$ means add to the $+^{ve}$ number 10 the $+^{ve}$ number 4. The result is the $+^{ve}$ number 14.

(2). Then $10+(-4)$ must mean add to the $+^{ve}$ number 10 the $-^{ve}$ number 4.

We know that whatever direction 10 is in, then −4 is in exactly the opposite direction.

Therefore 10 and −4 together give 6 in the direction of 10. That is +6.

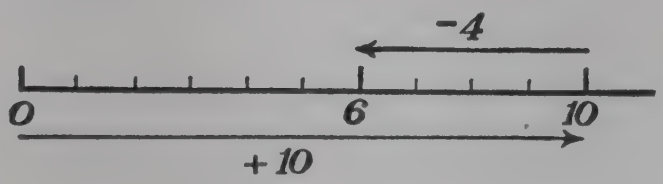


FIG. 1.

Therefore we can say $10+(-4)$ gives $10-4$ or 6.

(3). Also $10-(4)$ means 4 subtracted from 10, that is 6.

(4). Now let us consider $10-(-4)$.

We have seen that $10+(-4)$ means *subtract* 4 from 10. As + and − mean the opposite of one another, it follows that $10-(-4)$ will mean *add* 4 to 10. That is

$$10-(-4)=10+4=14.$$

12. When we particularly want a set of numbers to be considered together, we put them in a bracket. For example, $40+(15-6)$ means that we have to add to 40, $15-6$, that is 9, giving 49 as the result.

Now consider $40-(15+6)$.

Here we have to take from 40, $15+6$, that is 21, giving 19 as the result.

13. When we work our sum it is sometimes convenient to take away the brackets, but we must be quite sure that, when we do so, we are not altering values.

Considering again $40-(15+6)$, we see that this means we are to take away from 40, 6 more than 15.

If we write $40-15$, we have not yet taken away all that we were asked to take away.

We must still take away 6.

This gives

$$\begin{aligned} &40-(15+6)\\ =&40-15-6\\ =&40-21\\ =&19. \end{aligned}$$

Now take $40-(15-6)$.

This sum means that we are to take away from 40, 6 less than 15.

If we write $40-15$, we have taken away 6 more than we were asked to take away. Therefore, to keep our value unaltered, we must add 6.

That is,

$$\begin{aligned} &40-(15-6)\\ =&40-15+6\\ =&46-15\\ =&31. \end{aligned}$$

In the same way—

EXAMPLE 1.—

$$\begin{aligned} &12-(7-3)\\ =&12-7+3\\ =&15-7\\ =&8. \end{aligned}$$

EXAMPLE 2.—

$$\begin{aligned} &36-(7-3)+(15-6)\\ =&36-7+3+15-6\\ =&54-13\\ =&41. \end{aligned}$$

EXAMPLE 3.— $3x-(x+6)+(2x+4)$
gives $3x-x-6+2x+4$
or $4x-2$.

From the above examples we see that the following rule has to be observed when simplifying expressions containing brackets :—

If the sign immediately preceding a bracket is a $+^{ve}$ one, then the signs remain unchanged when the bracket is removed.

If the sign immediately preceding a bracket is a $-^{ve}$ one, then every sign in the bracket is altered when the bracket is removed.

Examples IIa

Lower.

Give the value of each of the following expressions, using the method of paragraph 13.

1. $15+(6+2)$.
2. $15-(6+2)$.
3. $15+(6-2)$.
4. $15-(6-2)$.
5. $7+(5+3)+(8-7)$.
6. $95-(20+4)-(16-8)$.
7. $32+(18-91)-(15-106)$.
8. $13x+5+(10x-3)$.
9. $6x-(2x+4)$.
10. $19+(3x+7)+(4x-8)$.
11. $13-(2-x)+(6x-4)$.

Solve the following equations, and check the answers.

12. $3x=2+(x+4)$.
13. $6x+8=x+9-(2x-6)$.
14. $2x-(3-6x)=16-(4x+3)+(24-2)$.
15. $5x+(8x-7)-(3+12x)=0$.
16. $17+5x+7-(x-10)=15+(3x-5)$.
17. $91-(82+3x)=x+(4x+16)-(11x-2)$.
18. $13x-(2x-15)+(3x-11)=12x-(2x-12)$.

Middle.

19. Give the value of $19-(7+6-14)$.

Solve the following equations :—

20. $18x-(21-3x)+(2x+7)=12-(x+4-2x)$.
21. $25+(3x+10-12)=5x-(4+6x-31)$.
22. $6x+(7-4x)-(3+2x-11)=3x-(5x-19)$.
23. $19+7x-(x-6)+11-4x=18-(7x-5)+(4x-2)$.

Higher.

24. $15x-(3-8x)-(5+6x)+(3x-8)=5x-(6-5x)+20$.
25. $4+(9x+7)-(4x-3)=7x-(3+11x)-1$.
26. $5x+4-(3x+5)-8x+(7x-3)=8-7x-(6+3x)-6$.
27. $19+x-(8-5x)+(9x-7)-3x=7-(2-5x)+3x+3$.

14. We have seen that the expression $2x$ means 2 multiplied by x. Now let us find a meaning for $2(x+12)$.

Consider a girl walking round the edge of a field which has sides x yd. and 12 yd. (fig. 2).

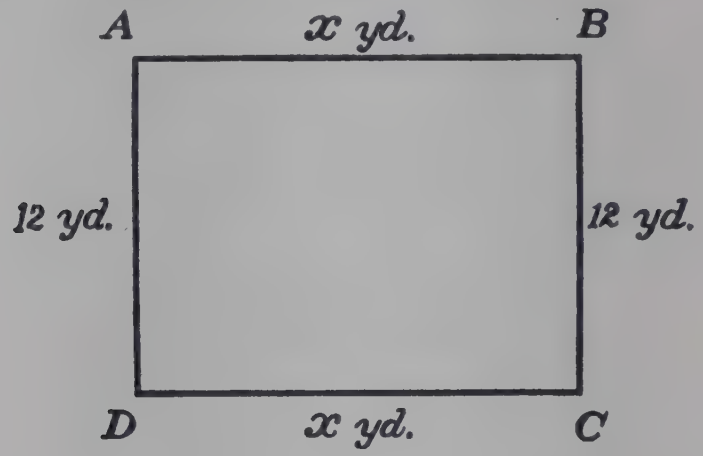


FIG. 2.

Suppose she starts at A and walks to B and then on to C. She will have walked half-way round the field—namely $(x+12)$ yd. Therefore the whole distance round the field must be $2(x+12)$ yd.

Adding the lengths of the four sides, we see that this distance is also $(x+12+x+12)$ yd., that is $(2x+24)$ yd.

$$\therefore\ 2(x+12)=2x+24.$$

That is, each term in the bracket has to be multiplied by the number outside the bracket.

EXAMPLE 1.—*Solve*

$$5x+3(x+2)=3x+2(x+6).$$

That is,

$$5x+3x+6=3x+2x+12$$
$$5x+3x-3x-2x=12-6$$
$$3x=6,$$
$$\therefore\ x=2.$$

EXAMPLE 2.—*Solve*

$$5x+3(4x+7)=9+4(x+1)+5(2x+7).$$
$$5x+12x+21=9+4x+4+10x+35$$
$$5x+12x-4x-10x=9+4+35-21$$
$$3x=27,$$
$$\therefore\ x=9.$$

Examples IIb

Solve the following equations :—

Lower.

1. $3(x+1)=2(x+2)$.
2. $5+2(x+4)=3+4(x+1)$.
3. $7x+5+3(x+7)=7(x+3)$.
4. $2x=3(x+5)-2x+3$
5. $6(2x+3)-10=2x+3(x+5)$.
6. $5(x+4)+6(2x+1)=3x+40$.
7. $3(2x+4)=6+5(x+10)$.
8. $3x+4(x+7)-(x+8)=5+(x-8)$.
9. $4x+3(x+8)-(3x+4)=0$.
10. $-9+5(x+3)+4(x+5)=7-(2x-37)$.

Middle.

11. $3x+12(5x+6)-(8-x)=5(3x+10)+14$.
12. $16+7x-(3x+11)+4(8+3x)=3x+5(2x+8)$.
13. $5x+7(3x+4)+4(5+9x)=14-(3x-7)+44x+6(x+12)$.

Higher.

14. $4x+8(3x+19)-(2x-5)+21=3(2x+7)+5(8+3x)+132$.
15. $65+11(5x+2)+4(x+7)=7x-(3x-80)+6(8x+7)$.

15. We are now used to the fact that, whatever process is represented by a number with a positive sign, the opposite process is represented by the same number with a negative sign.

Thus, if $3 \times x = 3x$
or $x \times 3 = 3x$, **A**

then $3 \times -x = -3x$
or $x \times -3 = -3x$, **B**

and $-3 \times -x = 3x$
or $-x \times -3 = 3x$. **C**

These results are rather strange, so we shall look at them again carefully.

The result in **C** is the same as in **A** although we begin with different terms. This is because there are two negative signs, so we have reversed the process twice which gives us the same result as if we had not reversed it at all.

Compare this with what would happen in gymnastics if you were given the commands "*right about turn*" (you face the opposite way to that you did before), and then "*right about turn*" again (you face the same way as you did at the start).

EXAMPLE 1.—*Simplify* $4x-3(x+2)$.

Here everything in the bracket has to be multiplied by -3.

This gives $4x-3\times x-3\times 2$
or $4x-3x-6$
or $x-6$.

EXAMPLE 2.—*Simplify* $11x-5(x-8)$.

This gives $11x-5\times x-5\times -8$
or $11x-5x+40$
or $6x+40$.

EXAMPLE 3.—*Solve*

$$4x+7(x-3)=2-3(x+3).$$
$$4x+7\times x+7\times -3=2-3\times x-3\times 3$$
$$4x+7x-21=2-3x-9$$
$$4x+7x+3x=2-9+21$$
$$14x=14,$$
$$\therefore\ x=1.$$

EXAMPLE 4.—*Solve*

$$5(x-2)-7(x+6)=2x-4(2x-7).$$
$$5x-10-7x-42=2x-8x+28$$
$$5x-7x-2x+8x=28+10+42$$
$$4x=80,$$
$$\therefore\ x=20.$$

Examples IIc

Lower.

Simplify the following expressions :—

1. $5x+6(x-4)$.
2. $5x-6(x-4)$.
3. $5x-6(x+4)$.
4. $7+2(3x-5)$.
5. $5(3x+4)-3(x+2)$.
6. $4(2x-7)-7(5-2x)-4x$.
7. $9+3(x-5)-4(6-5x)+11(x-3)$.
8. $5x-9(2x-3)-(x-7)+2(x+6)$.
9. $6x+32-4(3x-11)+17(x+1)$.
10. $40(x+3)-6(3x+1)-5(7-2x)$.

Solve the following equations, and check their answers :—

11. $3-2(x+1)=1-3x$.
12. $3(x+2)-4(x+1)=0$.
13. $5(3x-7)-(2x-10)=6x+10$.
14. $16+3x-7(2x+1)=7x-5(x+6)$.
15. $22x+7(3x+2)-41+6(x-4)=55-2(x+2)$.

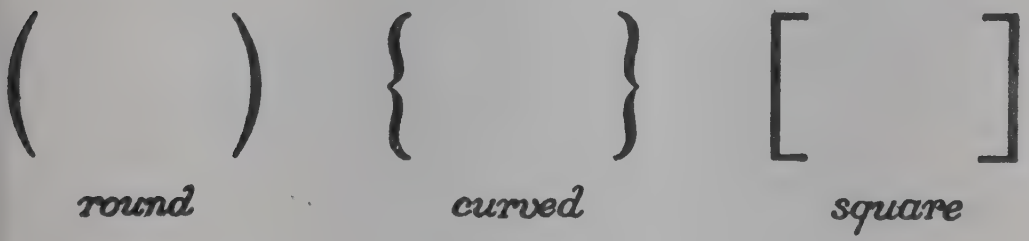
Middle.

16. $4(3x-7)+5(2x+11)-(x-10)=3-6(2x-13)-11.$
17. $19(2x+1)-7(3x-7)+8(x+9)=14(2+5x)-4(3x+5).$
18. $3(2x+5)-91+7(13-6x)-9x$
$=15x+10(x+2)-5(2x+7).$
19. $4(9+3x)-5(2x-7)+16(8-2x)$
$=12(3x+7)-10(x+4)-13.$

Higher.

20. $3x+5(8-4x)+9(3x-11)$
$=6(3x+7)+5(2x-10)-11(2x-7).$
21. $14+2(3x-5)=95-7(2x-5)-3(x-4).$
22. $27-3(12-5x)-9(2x+7)=15(x+2)+12(3x-4).$
23. $4(3x+11)-7(2x+8)+19x=4-(4x+9)+7(2x-5).$

16. When we have more than one set of brackets round certain numbers, we make each pair a different shape so that we can easily see which expression is enclosed in each particular bracket, *e.g.*—



Sometimes we put a line over terms we wish to consider as being in a bracket,

e.g. $x+\overline{b+c}.$

This line is called a *Vinculum*. You already know it quite well from fractions where we write $\frac{12\frac{1}{2}}{7-\frac{1}{4}}$, which means that $7-\frac{1}{4}$ is to be considered as in one bracket and $12\frac{1}{2}$ is to be divided by the result.

17. When there are a number of brackets in one sum, you must be careful how you remove them.

Do this step by step, beginning with the inside bracket and taking care that the value of the result, when all the brackets have been removed, is the same as the original value.

EXAMPLE 1.—*Solve* $60x-11\{3x+(2x+4)\}=26$.

Removing the inside bracket, we have

$$60x-11\{3x+2x+4\}=26.$$

Removing curved bracket, we have

$$\begin{aligned} 60x-33x-22x-44&=26\\ 5x&=26+44\\ 5x&=70\\ x&=14. \end{aligned}$$

EXAMPLE 2.—*Simplify* $x+8a-[7x+\{2b-(3x+c+4a)\}]$.

The expression

$$\begin{aligned} &=x+8a-[7x+\{2b-3x-c-4a\}]\\ &=x+8a-[7x+2b-3x-c-4a]\\ &=x+8a-7x-2b+3x+c+4a\\ &=-3x+12a-2b+c. \end{aligned}$$

EXAMPLE 3.—*Solve*

$$7x+[3x-\{2x+4(6x-\overline{8+2x})\}+20]=30.$$

Removing Vinculum :—

$$\begin{aligned} 7x+[3x-\{2x+4(6x-8-2x)\}+20]&=30\\ 7x+[3x-\{2x+24x-32-8x\}+20]&=30\\ 7x+[3x-2x-24x+32+8x+20]&=30\\ 7x+3x-2x-24x+32+8x+20&=30\\ 7x+3x-2x-24x+8x&=30-32-20\\ -8x&=-22\\ 8x&=22\\ x&=2\tfrac{3}{4}. \end{aligned}$$

Examples IId

Lower.

Simplify the following expressions :—

1. $32+2\{2x+3(x-7)\}$.
2. $5+4\{3x-7(x+6)+4(2x-3)\}+5x$.
3. $8x+6+\{4+3(x-9)-7(5x-\overline{2x-3})\}$.
4. $2+[7x+\{x+5(x-2)\}+9]$.
5. $3x+2[x-\{5+3x-4(3x-2)\}]$.

Solve the following equations :—

6. $3\{3x+(x+2)\}=78$.
7. $7\{x+12(3x+7)-4(x+4)\}=1+10(32x+3)$.
8. $5x-[3x+1-\{2x+1-6(2x+3)\}]=4-30x$.
9. $3+2x-\{2x-4+(4x-7-\overline{3x-3})\}=8$.

Middle.

10. $2\{x+[3x-5(2x-\overline{7x+6})]\}=2(3x+4)+74.$
11. $6x-35+5[3-4(7+3x)]=4(x+3)+7[x-2(x+5)].$
12. $5-3\{x-6[4-(3x+7)]+8(x-2)\}=170.$

Higher.

13. $4x-7\{3x+5(x-4)-3[x-(5-3x)]\}=2(x+10)+35.$
14. $3(2x+1)+7\{3-5(x-4)+3[x-6(2x-1)]\}=30.$
15. $9x-7\{3x-5(2x+7)\}+4(x-5)-23=0.$

18. So far we have only considered equations containing whole numbers. Sometimes it is necessary to solve equations containing fractions, such as :—

EXAMPLE 1.— $$\frac{x}{3}+\frac{x}{2}=\frac{x}{6}+12 \quad . \quad . \quad . \quad (1)$$

It is not, as a rule, convenient to keep fractions in equations. The first step is, therefore, to multiply both sides by the smallest number into which each of the denominators will divide exactly.

In equation (1) we have to find the L.C.M. of 3, 2, and 6. This is 6.

Multiplying both sides by 6, we have

$$6\left(\frac{x}{3}+\frac{x}{2}\right)=6\left(\frac{x}{6}+12\right);$$

i.e.
$$\begin{aligned} 2x+3x&=x+72\\ 2x+3x-x&=72\\ 4x&=72,\\ \therefore\ x&=18. \end{aligned}$$

EXAMPLE 2.—*Solve* $\dfrac{x+3}{4}+\dfrac{x-2}{7}=\dfrac{2x-8}{5}+2.$

Multiply both sides by 140.

$$\begin{aligned} 140\left(\frac{x+3}{4}+\frac{x-2}{7}\right)&=140\left(\frac{2x-8}{5}+2\right)\\ 35(x+3)+20(x-2)&=28(2x-8)+280\\ 35x+105+20x-40&=56x-224+280\\ 35x+20x-56x&=-224+280-105+40\\ -x&=-9\\ x&=9. \end{aligned}$$

EXAMPLE 3.—*Solve* $\frac{3}{4}(x+2)+\frac{1}{8}(2x-3)=2x-\frac{7}{8}$.

Multiply both sides by 8.

$$6(x+2)+(2x-3)=16x-7$$
$$6x+12+2x-3=16x-7$$
$$6x+2x-16x=-7-12+3$$
$$-8x=-16$$
$$8x=16$$
$$x=2.$$

EXAMPLE 4.—*Solve* $\frac{3x+2}{2}-\frac{3}{4}x-\frac{3x-8}{2}=2$.

Multiply both sides by 4.

$$2(3x+2)-3x-2(3x-8)=8$$
$$6x+4-3x-6x+16=8$$
$$6x-3x-6x=8-4-16$$
$$-3x=-12$$
$$3x=12$$
$$x=4.$$

Examples IIe

Solve the following equations :—

Lower.

1. $\frac{x}{3}+\frac{x}{4}=7$.
2. $\frac{2x}{3}+\frac{x}{2}-\frac{5x}{6}=2$.
3. $\frac{27x}{5}+\frac{x}{3}=\frac{2x}{3}+\frac{76}{5}$.
4. $\frac{x+1}{6}+\frac{2x-1}{9}=2$.
5. $\frac{3x+5}{7}-1=\frac{x+3}{6}$.
6. $\frac{6x-5}{5}-\frac{3x-5}{2}+\frac{3}{5}=0$.
7. $\frac{3x}{5}+\frac{x+2}{6}=x-2$.
8. $\frac{3x}{4}+\frac{x+8}{3}-\frac{x}{15}=4\frac{7}{10}$.
9. $\frac{5}{6}(x+3)-\frac{2}{27}(8x+3)=3x-6$.
10. $\frac{2}{3}(5x+2)=\frac{4}{7}(20-3x)$.
11. $\frac{5(13-4x)}{8}=\frac{3(2x-1)}{24}$.
12. $\frac{1}{6}x-\frac{2}{3}(3x+5)=x+\frac{1}{6}-\frac{7x-5}{2}$.
13. $\frac{x}{3}-\frac{2x-3}{9}+\frac{4+5x}{2}=18$.
14. $\frac{2x+5}{7}+\frac{20-2x}{4}+\frac{3}{7}(5x+2)=22$.
15. $\frac{3x+5}{5}+\frac{7x}{2}-\frac{2x+8}{7}=38$.

Middle.

16. $\frac{24-x}{3}-\frac{3}{4}(3x+4)+25=11-x.$

17. $\frac{1}{2}(3x+5)-10+\frac{2}{3}(4x-7)=12\frac{1}{2}+\frac{9}{2}(x-6)$

18. $4(x+9)-\frac{2}{5}(3x+7)+\frac{5}{6}(2x-7)=31\frac{5}{6}.$

Higher.

19. $\frac{2x+9}{5}+\frac{x-2}{8}+\frac{7}{40}(9-x)=2\frac{3}{40}.$

20. $5(2x+3)-\frac{4x+5}{3}+\frac{2-12x}{2}=\frac{31}{6}(x+1)-\frac{5}{6}.$

Examples IIf

Miscellaneous Problems

Lower.

1. A father is four times as old as his son. In five years' time he will be three times as old as his son. What is his present age?

2. Doris and Hilda bought some apples. Doris bought 8 lb. more than Hilda. If Hilda had bought 3 lb. more, she would have bought half as many pounds as Doris. How many pounds did they each buy?

3. John has 40 marbles and Harry has 8 marbles. How many must John give Harry in order that he may have three times as many as Harry?

4. The sum of the third, fifth, and seventh parts of a number is 71. Find the number.

5. Coffee costs 6d. per lb. more than tea. If 6 lb. of tea and 7 lb. of coffee cost £2, 2s. 6d., find the cost of 1 lb. of each.

6. Two men together buy a house costing £1400. If thirteen times the amount paid by one man is equal to fifteen times the amount paid by the other, find the amounts paid by each.

7. The sum of the seventh, eighth, and tenth parts of a number is 206. Find the number.

8. Two numbers differ by 20. One seventh of the larger added to one-half of the smaller gives 44. Find the numbers.

9. I have £1, 15s. 0d. in shillings and half-crowns. I have 23 coins altogether. How many have I of each kind?

10. Find two consecutive numbers such that six times the smaller added to nine times the greater is 114.

11. Sixty marbles are divided between four boys, Jack, John, Harry, and William. John has twice as many as Jack,

Harry has 10 more than John, and William has 5 more than Harry. How many has each ?

12. The sum of two numbers is 105. One-third of the smaller number and four times the larger number together equal 310. Find the numbers.

Middle.

13. A number consisting of 2 digits (that is, of two figures) is such that the unit's digit is 3 less than the ten's digit. The number is 3 less than seven times the sum of the digits. Find the number.*

14. A messenger starts at noon from Liverpool and travels at 20 miles an hour. Two hours later a second messenger starts and travels along the same road at 40 miles an hour. How long will it take him to catch up the first messenger, and how far will they be then from Liverpool ?

Higher.

15. A number consisting of 2 digits is such that its unit's digit is one-third of its ten's digit. The sum of three times the unit's digit and five times the ten's digit is equal to 39 less than the number. Find the number.

16. The Guide funds amount to £3, 8s. 8d., and consist of half-sovereigns, florins, and pence. The total number of coins is 72, there being three times as many florins as there are half-sovereigns. Find the number of pence.

Examples IIg (*revision*)

Solve the following equations :—

1. $12x+3\frac{1}{2}-7x+15=22\frac{1}{2}-14x.$
2. $3x-7(2x-4)+5(5-3x)=11(x-10)+3x.$
3. $15x+3\{5x-4(3-2x)+7(5-2-3x)\}=0.$
4. $\frac{x}{3}-\frac{2x+8}{9}=\frac{4x-1}{5}-\frac{3x-1}{15}.$
5. $14{\cdot}7x+81{\cdot}5-3{\cdot}7x=13x+100.$
6. $\frac{5x-9}{4}-\frac{3x-61}{7}=\frac{2}{7}(4x-1)+\frac{3x}{4}$

* *Consider a number which has x for its ten's digit and y for its unit's digit. How shall we write it ?*

We know that the number 37 *means* $(3\times10)+7$.

That is, we have to multiply the ten's digit by 10 *and then add the unit's digit.*

Therefore the number we are considering may be written $10x+y$.

7. $8[\{3x-(x-8)+2(3x-7)\}-21]=4(x+2)$.
8. $91x-33+54x-27+14+21\cdot5x-11\cdot5x=0$
9. $\frac{2x+5}{3}-\frac{4(3x-7)}{5}+\frac{3(2x-1)}{5}=\frac{4(2x+1)}{15}$.
10. $8x-21+57x-63\frac{1}{2}+12\frac{5}{6}-14x=80\frac{2}{3}-4x$.
11. $14x-210+3x-81x+19=28+65-50x$.

Simplify :—

12. $4[3x-7\{2x+18-5(4x-1)\}]$.
13. $\frac{3x+4}{11}-\frac{2x-5}{3}+\frac{8(2x+7)}{33}$.

14. Two girls, Nita and Hilda, bought some material at 3s. and 2s. per yd. respectively. Together they bought 12 yd. If Hilda spent 1s. less than Nita, how many yards did they each buy ?

19. When we began algebra, we used it to solve problems. We found that, if we were told facts about a number, it was often possible to write these facts in the form of an equation and, from this equation, to find out the number. Our first problem was the very simple one of finding out how many apples there were in a bag, having been told that, if there had been three times as many apples, their number would have been thirty.

From the facts we were told we wrote down the very simple equation—

$$3x=30,$$
$$\therefore\ x=10.$$

As we have done more algebra, we have been able to solve more difficult problems, but our method has always been the same. We have let x represent the unknown quantity, and have written down the facts told us about this quantity in the form of an equation.

Let us consider Problem 4, Example II f. Here there is an unknown quantity, the number we are asked to find: therefore we have no difficulty in deciding to let x represent it. We then consider the facts we are told about x, and write these facts down in the form of the following equation :—

$$\frac{x}{3}+\frac{x}{5}+\frac{x}{7}=71.$$

We solve this equation and find that the unknown number is 105.

Now let us consider Problem 9 in the same set of examples. We see at once that this is not so simple. To begin with, there are two unknown quantities and we have to decide which shall be represented by x. When we have decided to let x represent the number of shillings, we find that, from the facts given us, the number of half-crowns may be written down as $23-x$. If we then express the value of the coins in shillings, we can write down the equation :—

$$x+2\tfrac{1}{2}(23-x)=35,$$

i.e. $$2x+5(23-x)=70,$$

and hence we find that $x=15$, from which we see that 15 of the coins were shillings and 8 were half-crowns.

20. This second problem may be solved in a slightly different way. There are two unknown quantities, namely, the number of shilling pieces and the number of half-crown pieces. Suppose we represent *each* unknown quantity by a letter—

e.g. let x represent the number of shilling pieces,
and let y ,, ,, ,, half-crown pieces.

We find that we are told enough about these two quantities to write down *two* equations, the first depending on what we are told about the number of the coins, and the second on what we are told about the value of the coins.

$$\begin{cases} x+y=23 & . \quad . \quad . \quad (1) \\ x+2\tfrac{1}{2}y=35 & . \quad . \quad . \quad (2) \end{cases}$$

We put a bracket round these two equations to remind us that the solution of our problem depends on *both* equations. We have to find values of x and y which will satisfy both equations.

In equation (2) we have the statement that two expressions are equal. If from the left-hand side of this equation we were to subtract the quantity $(x+y)$, we should have to subtract an *equal* amount from the right-hand side in order to keep an equation satisfying the same condition.

What amount is equal to $x+y$?

We see from (1) that $23=x+y$.

Hence $(x+2\frac{1}{2}y)-(x+y)=35-23$, . . (3)

i.e. $x+2\frac{1}{2}y-x-y=12$,

i.e. $1\frac{1}{2}y=12$,

an equation containing one unknown quantity which we can solve. This gives $y=8$.

Equation (3), from which we get this value of y, depends on both (1) and (2), *i.e.* the value $y=8$ satisfies both (1) and (2) if it satisfies (3).

Therefore if we write 8 in place of y in (1), the left-hand side of the expression will still equal the right-hand side,

i.e. $x+8=23$

$\therefore\ x=15$.

From this we see that we have found the value of x, which gives the number of shilling pieces, and of y which gives the number of half-crown pieces, these values being $x=15$, $y=8$.

Our result then is, there were { 8 half-crown pieces. / 15 shilling pieces.

We see that this gives the required total £1, 15s. 0d.

21. Let us consider another problem: *Two towns* A *and* B *are* 34 *miles apart. If a man starts from* A *at* 9 *a.m. on a motor cycle, and a boy leaves* B *at* 9 *a.m. on a bicycle, they meet at* 10 *a.m. If the man increases his average rate in miles per hour by* 2, *he will travel three times as quickly as the boy. At what rate did they travel?*

Let x represent the boy's rate in m.p.h. (miles per hour).

„ y „ „ man's „ „

In 1 hour they cover the whole distance between A and B,

$\therefore\ x+y=34$ . . . (1)

If the man had gone 2 m.p.h. faster, *i.e.* $(y+2)$ m.p.h., he would have gone three times as quickly as the boy; that is at $3x$ m.p.h.,

$\therefore\ 3x=y+2$ . . . (2)

i.e. our problem gives rise to the two equations:—

$$\begin{cases} x+y=34 & . \ . \ . \ \text{(i)} \\ 3x-y=2 & . \ . \ . \ \text{(ii)} \end{cases}$$

If we add the left-hand side of equation (2) to the left-hand side of equation (1), the resulting expression will equal 34+2,

$$\therefore\ x+y+(3x-y)=34+2 \quad . \quad . \quad (3)$$

an equation depending on both (1) and (2), which gives

$$4x=36,$$
$$\therefore\ x=9.$$

Hence, from (1) if $x=9$,

$$9+y=34,$$
$$\therefore\ y=25.$$

The condition that (1) and (2) are both satisfied is, therefore,

$$\begin{cases} x=9. \\ y=25. \end{cases}$$

i.e. the man rides at 25 m.p.h. and the boy rides at 9 m.p.h.

We can verify or check our results and see that these values satisfy the statements that are made in the problem.

EXAMPLE 1.—*On Saturday a boy spends* 2*s.* 2*d. of his pocket-money on half-a-pound of chocolates and a quarter of a pound of caramels. The next week he spent* 2*s.* 4*d., and bought half-a-pound of caramels and a quarter of a pound of chocolates, but the latter cost* 3*d. more per quarter lb. than those he had the previous week. What was the price per quarter lb. of the caramels and of the dearer kind of chocolates ?*

Let xd. be the price per quarter lb. of the caramels.

,, yd. ,, ,, ,, first chocolates.

Then $(y+3)$d. is the price per quarter lb. of the second chocolates.

(1st Saturday)

$x+2y=26$. (2s. 2d. expressed in pence=26d.).

(2nd Saturday)

$2x+(y+3)=28$. (2s. 4d. ,, ,, =28d.);

i.e.

$$\begin{cases} x+2y=26 \quad . \quad . \quad . \quad (1) \\ 2x+y=25 \quad . \quad . \quad . \quad (2) \end{cases}$$

As these equations stand at present, neither by adding nor by subtracting could we get from them an equation

containing only one unknown, but if we multiply each term in (1) by 2 we get

$$2x+4y=52 \quad . \quad . \quad . \quad . \quad (3)$$

Then, if we subtract (2) from (3) we get

$$2x+4y-(2x+y)=52-25;$$

i.e.

$$4y-y=27$$
$$3y=27$$
$$y=9.$$

Hence from (1) if

$$y=9$$
$$x+18=26$$
$$x=8.$$

The solution to our equations is, therefore,

$$\begin{cases} x=8 \\ y=9. \end{cases}$$

The cost of the caramels is, therefore, 8d. per quarter lb.: the first chocolates cost 9d. per quarter lb., and the second chocolates cost 1s. per quarter lb.

22. When we have *one* problem from which we can build up two equations, each with two unknown quantities, we give a special name to the two equations. They are called *Simultaneous Equations*. This means that the *same* values for the unknown quantities satisfy *both* equations. Every problem containing two unknown quantities can be solved by simultaneous equations; when the problem is very simple, we sometimes do not need two equations, but can write down the facts we are told in the form of one equation. For example, see Examples Id and Examples IIf. If the problem is more difficult, we use two letters, one to represent each unknown quantity, and write down the facts we are told in the form of two equations. From these *two* equations we try to find *one* equation containing one unknown quantity. We "get rid of" one unknown either by addition or subtraction, and we must first see that our equations are in a form which will enable us to do this.

"The getting rid" of one unknown is called **eliminating** it. To eliminate is a long word, which means to cut or to cancel.

EXAMPLE 1.—*If 3 lb. of tea and 4 lb. of coffee cost* £1, *and 5 lb. of tea and 7 lb. of coffee cost* £1, 14*s.* 4*d.*, *what is the price per lb. of tea and coffee?*

Let xs. be the price of 1 lb. of tea.
„ ys. „ „ 1 „ coffee.

Write the cost of 3 lb. of tea and 4 lb. of coffee in shillings.

This gives $3x+4y=20$.

Write the cost of 5 lb. of tea and 7 lb. of coffee in shillings.

This gives $5x+7y=34\frac{1}{3}$;

i.e.
$$\begin{cases} 3x+4y=20 & . \quad . \quad . \quad (1) \\ 5x+7y=34\frac{1}{3} & . \quad . \quad . \quad (2) \end{cases}$$

These are the two simultaneous equations, and from them we are to try to find one equation by eliminating either x or y.

If we decide to eliminate x, we must arrange the equations so that we have the same multiple of x in both equations.

Let us multiply (1) by 5 and (2) by 3.

We then have
$$\begin{cases} 15x+20y=100 & . \quad . \quad . \quad (3) \\ 15x+21y=103 & . \quad . \quad . \quad (4) \end{cases}$$

We can now subtract (4) from (3) without rewriting them:

$$-y=-3;$$

i.e.
$$y=3.$$

From (1) if $y=3$, then $3x+12=20$
$$3x=8$$
$$x=2\tfrac{2}{3}.$$

Our equations are both satisfied if $x=2\frac{2}{3}$ and $y=3$. That is, the tea costs 2s. 8d. per lb., and the coffee costs 3s. per lb.

23. Let us consider the arrangement of our work after the stage when we have found the simultaneous equations.

Suppose we have a problem which can be stated in equation form as follows:—

$$\begin{cases} 3x+7y=-1 & . \quad . \quad . \quad (1) \\ 2x-8y=12 & . \quad . \quad . \quad (2) \end{cases}$$

To eliminate x, multiply (1) by 2 and (2) by 3 and subtract.

Then
$$\begin{cases} 6x+14y=-2 & . \quad . \quad . \quad (3) \\ 6x-24y=36 & . \quad . \quad . \quad (4) \end{cases}$$
$$38y=-38,*$$
$$y=-1.$$

If $y=-1$, from (1) we get
$$3x-7=-1$$
$$3x=-1+7$$
$$3x=6$$
$$x=2,$$

i.e. $x=2$, $y=-1$ are the only values which satisfy both equations
$$\begin{cases} 3x+7y=1 & (1) \\ 2x-8y=12 & (2) \end{cases}$$

Or again, if the simultaneous equations to be solved are
$$\begin{cases} 7x+2y=1 & . \quad . \quad . \quad (1) \\ 11x-y=43 & . \quad . \quad . \quad (2) \end{cases}$$

To eliminate y multiply (2) by 2 and add. Then
$$\begin{cases} 7x+2y=1 & . \quad . \quad . \quad (1) \\ 22x-2y=86 & . \quad . \quad . \quad (3) \end{cases}$$
$$29x=87$$
$$x=3.$$

If $x=3$, then from (2) $33-y=43$
$$-y=43-33$$
$$-y=10$$
$$y=-10.$$

i.e. $x=3$, $y=-10$ are the values of x and y which satisfy both equations
$$\begin{cases} 7x+2y=1 & (1) \\ 11x-y=43 & (2) \end{cases}$$

24. Simultaneous equations may also be solved as follows :—

Solve
$$\begin{cases} 3x+5y=21 & . \quad . \quad . \quad (1) \\ x+4y=14 & . \quad . \quad . \quad (2) \end{cases}$$

* *When we subtract* (4) *from* (3) *we have*

$$6x+14y-(6x-24y)=-2-36 \quad . \quad . \quad . \quad (A)$$
or $$6x+14y-6x+24y=-2-36 \quad . \quad . \quad . \quad (B)$$
i.e. $$38y=-38.$$

We may omit lines (A) *and* (B) *if we remember that, in subtracting* (4), *we should have the negative sign before the whole of its left-hand and right-hand expressions.*

From equation (1) we have

$$5y=21-3x$$
$$y=\frac{21-3x}{5}.$$

Similarly from equation (2) we have

$$4y=14-x$$
$$y=\frac{14-x}{4}.$$

If y is to have the same value in both equations, in this special case we have

$$\frac{21-3x}{5}=\frac{14-x}{4}.$$

This is an equation with only one unknown, which we may solve in the usual way—

$$\frac{21-3x}{5}=\frac{14-x}{4}.$$

Multiply throughout by 20—

$$4(21-3x)=5(14-x)$$
$$84-12x=70-5x$$
$$-12x+5x=70-84$$
$$-7x=-14$$
$$x=2.$$

In (i) when $x=2$ $y=\frac{21-6}{5}=\frac{15}{5}=3.$

In (ii) ,, $x=2$ $y=\frac{14-2}{4}=\frac{12}{4}=3.$

This is as we expected. We have found the one case when x and y have the same values in both equations ; that is, we have solved the equations.

$\left.\begin{matrix}x=2\\y=3\end{matrix}\right\}$ is the only condition that $\left\{\begin{matrix}3x+5y=21 & (1)\\ x+4y=14 & (2)\end{matrix}\right.$ are both true.

Examples IIh

Solve the following simultaneous equations :—

Lower.

1. $\begin{cases} x+4y=14. \\ x+\ y=5. \end{cases}$
2. $\begin{cases} 3x+2y=17. \\ x+2y=7. \end{cases}$
3. $\begin{cases} 7x-4y=19. \\ 2x+4y=26. \end{cases}$
4. $\begin{cases} 3x-2y=22. \\ 3x+7y=4. \end{cases}$
5. $\begin{cases} 2x-7y=15. \\ 5x+7y=-36. \end{cases}$
6. $\begin{cases} 2x+5y=43. \\ 4x+9y=79. \end{cases}$
7. $\begin{cases} 7x+3y=29. \\ 4x-9y=38. \end{cases}$
8. $\begin{cases} 8x+5y=-47. \\ 4x-15y=29. \end{cases}$
9. $\begin{cases} 5x+4y=38. \\ 3x+7y=55. \end{cases}$
10. $\begin{cases} 7x-3y=50. \\ 3x+7y=38 \end{cases}$

Middle.

11. $\begin{cases} \frac{1}{2}x+y=5. \\ \frac{1}{2}x-y=-1. \end{cases}$
12. $\begin{cases} \frac{3}{2}x+\frac{5}{6}y=19. \\ \frac{3}{2}x+\frac{1}{6}y=11. \end{cases}$
13. $\begin{cases} 2x=5y-7. \\ 5x=8y-13. \end{cases}$
14. $\begin{cases} 3x+4y=-3. \\ 2y-5x=-21. \end{cases}$

Higher.

15. $\begin{cases} 0{\cdot}3x+2y=13. \\ 0{\cdot}6x+5y=31. \end{cases}$
16. $\begin{cases} 0{\cdot}5x-0{\cdot}3y=-1. \\ 1{\cdot}2x+0{\cdot}5y=-0{\cdot}2. \end{cases}$
17. $\begin{cases} 3x-2y=-\frac{1}{6}. \\ 5x+4y=5\frac{5}{6}. \end{cases}$
18. $\begin{cases} \frac{1}{2}x+\frac{2}{3}y=14\frac{1}{2}. \\ \frac{2}{3}x-\frac{1}{2}y=-5\frac{2}{3}. \end{cases}$

Examples IIi

Use simultaneous equations to solve the following problems.

1. The sum of two numbers is 31, and their difference is 3. Find the numbers.

2. A father is four times as old as his son. In twenty years' time he will be twice as old as his son. What are their present ages ?

3. Two pounds of red currants and one pound of raspberries can be bought for 2s. 9d. A pound and a half of each can be bought for 2s. $10\frac{1}{2}$d. What is the price of each per pound ?

4. It costs 3 guineas a day to employ 5 men and 3 women, and £2, 17s. a day to employ 5 women and 3 men. At what rate per day is each paid ?

5. If 1 be added to the numerator of a certain fraction it equals $\frac{2}{3}$. If 2 be added to the denominator, the fraction equals $\frac{1}{2}$. Find the fraction.

6. If 1 be added to both the numerator and the denominator of a certain fraction the result is $\frac{4}{5}$. If 1 be subtracted from both numerator and denominator the result is $\frac{3}{4}$. Find the fraction.

7. I have £1, 10s. in florins and half-crowns. I have 13 coins altogether. How many are florins ?

8. I have £3, 8s. in shillings and florins. I have 44 coins altogether. How many are shillings ?

9. If 3 lb. of apples and 4 lb. of nuts together cost 6s. whilst 5 lb. of apples and 3 lb. of nuts cost 6s. 4d., what are the prices of nuts and apples per lb. ?

10. Divide £2 into two parts, so that twice the smaller part may be 1s. 6d. less than the larger part.

11. Divide £12, 5s. between two girls, so that for every shilling one receives, the other receives half a crown.

12. Tom and Harry have some marbles. If Tom were to give Harry 15, they would have the same number each ; if Harry gave Tom 10, Tom would have six times as many as Harry. How many has Tom ?

13. Two digits forming a number change places if 9 is added to the number, and the sum of the new number and twice the original number is 111. Find the number.

14. If 30 lb. of strawberries and 40 lb. of raspberries be together worth £3, 17s. 6d. and 36 lb. of strawberries and 70 lb. of raspberries be together worth £5, 15s., what is the value of 5 lb. of strawberries and 6 lb. of raspberries ?

15. Four lb. of tea and 4 lb. of chicory together cost 12s. If chicory were to cost half as much again and tea were to rise 3d. per lb., the same quantities would cost 14s. Find the present price of the tea per lb.

16. The prices of cheese and butter are such that $2\frac{1}{2}$ lb. of cheese and 4 lb. of butter cost 11s., and if the price of cheese were to decrease by 2d. a lb. and the price of butter were to rise 2d. a lb., then 16 lb. of cheese and 10 lb. of butter would cost £2, 6s. Find the price of butter and cheese per lb.

CHAPTER III

Geometry

25. You have done a considerable amount of work in practical geometry ; do you remember some of the main facts you discovered about geometrical figures ?

1. **If one straight line stands on another straight line, the sum of the two adjacent angles thus formed is two right angles or 180 degrees.**

2. **If, from a point in a straight line, two other straight lines are drawn, one on each side of it, and the sum of the two angles thus formed is 180 degrees, then the two straight lines make one straight line.**

3. **If you wish to make a triangle of a definite size and shape, there are certain sets of facts, one set of which you must be told—**

(*a*) **The lengths of the three sides,**
or (*b*) **the lengths of two sides and the size of the angle between them,**
or (*c*) **the length of one side and the size of two angles,**
or (*d*) **in the important case of the right-angled triangle, the lengths of any two sides may be given.**

4. (*a*) **If two parallel straight lines are cut by another straight line** (*a transversal*), **the corresponding angles are equal.**

(*b*) **If a straight line crosses two other straight lines and the corresponding angles are equal, then the two straight lines are parallel to one another.**

We shall for the present refer to the above facts as **assumptions,** because, though they can be proved to be true, we intend to make use of the facts before we give the proofs.

26. We have used the word **proved** here. What exactly do we mean by it ?

A proof is a method of reasoning which one's mind accepts as completely reliable. It is clear that we must be very careful about the methods we use when trying to give a proof. First of all, we must realise that we cannot rely on the evidence of our senses unless our minds actively agree with it. It is possible to hold a penny piece before our eyes in such a way that it blots out a large building like St Paul's Cathedral. To the eye, then, the penny would appear the larger, but the mind **knows** that there is some other explanation. Again, we are all familiar with the fact that, as one looks along a straight railway track, the lines appear to meet. If we depended solely upon the evidence of our eyes, we should say that the lines do meet, but a very limited knowledge of perspective drawing enables the mind to check the impression received from the eyes.

Draw any straight line and measure it in centimetres. Leave your ruler in position and move your head about six inches to the left of the line. Now read off the measurement of the line. Repeat this with a movement of your head to the right. You will now be able to realise to what extent the accuracy of any measurement is liable to be affected by the least carelessness in the measurer.

Are you good at ear tests ? If so, you will realise how much better, as an instrument of measurement, your ear is than those of some of your acquaintances, but there are some people who can judge sounds with yet a greater degree of accuracy than you can.

If you were given two packages, one weighing $2\frac{1}{2}$ lb. and the other 2 lb. 7 oz., could you be certain, by feeling the weight, which was the 2 lb. 7 oz. packet ?

27. These examples are given to impress on you the fact that your senses are not always to be relied upon as guides ; they may completely deceive you or only very approximately inform you. It is the sense of sight that we use in our geometry, and we must test its evidence in the manner referred to in the preceding paragraph. We draw a figure to help us in our work and, if we are sensible, study it carefully to get from it as much help as possible. What we see may be a valuable guide to us in our proof, but it

will not be the proof which must depend on something more definite and accurate than, for example, that two things **look** alike or unalike.

28. To base our proofs upon results obtained by measurement is unsatisfactory in another respect, for whilst the results obtained give us information about the particular figure or figures we have considered, they give us no ground for conclusion about any other figure.

It is not only in geometry that we have to guard against stating general conclusions when our knowledge is based upon observation of particular cases only. Suppose we found out that the majority of the members of five tennis teams preferred grass to hard courts, we could not, without inquiry, be certain that a majority in every tennis team would also be in favour of grass courts.

When we have something to prove about the sides or angles of a triangle, we must consider not only the particular triangles we have drawn, but the general nature of triangles, and our proof must depend upon general facts that we can state as being always true.

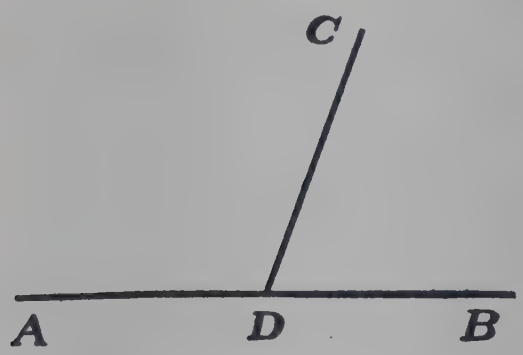
If we consider the age of a pupil as compared with that of his teacher, we may be able to say that, in every case within our knowledge, the teacher is older than his pupil. At the same time we realise that, among all the teachers and pupils whom we do not know, particularly among those at the classes in the evening institutes, it is quite possible for a pupil to be older than his teacher.

But, when we say that a mother is older than her child, we know that we are stating a completely general truth to which, in the nature of things, there can be no exception. And it is with **completely** general truths about the nature of geometrical figures, and not with the results obtained from the measurement of particular ones, that we are now chiefly concerned.

Measurement is not reliable, measurement is not accurate, measurement is not general. The eye can be deceived; in a geometrical proof we appeal to the mind. It is longer but safer.

29. EXAMPLE 1.—*If a straight line* **CD** *stands on a straight line* **ADB** *and* $\angle$**CDB**$=80°$, *how many degrees are there in* $\angle$**CDA** ?

EXAMPLE 2.—*If a straight line* **CD** *stands on a straight line* **ADB** *and* $\angle$**CDA** *is* $40°$ *greater than* $\angle$**CDB**, *what is the size or magnitude of these two angles ?*



We use algebra here.

Let $\angle$**CDB**$=x°$.

Then $\angle$**CDA**$=x°+40°$.

We know that

$x°+x°+40°=180°$ (from Assumption 1);

i.e. $2x+40=180$,

i.e. $2x=140$,

i.e. $x=70$,

i.e. $\angle$**CDB**$=70°$,

and $\angle$**CDA**$=70°+40°$

$=110°$.

To satisfy the given conditions, $\angle$**CDB**$=70°$ and $\angle$**CDA**$=110°$.

30. In the figure of *Example 2*, paragraph 29, $\angle$**CDA** and $\angle$**CDB** have the same vertex **D** and lie on opposite sides of a common arm **DC**, *i.e.* they are **adjacent.** But the two angles of our figure are not only adjacent ; together they make up $180°$, or half a revolution. Each supplies the other with the number of degrees necessary to bring the total to $180°$. Such angles are called **supplementary** to one another.

Are two adjacent angles necessarily supplementary to one another ? Illustrate your answer by a figure.

Examples IIIa

Lower.

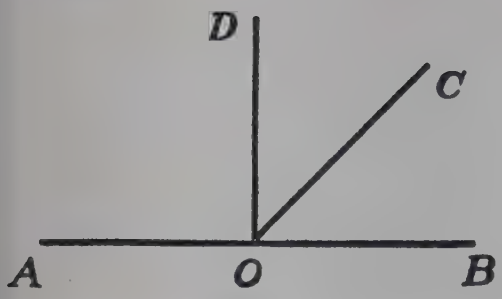


Fig. 1.

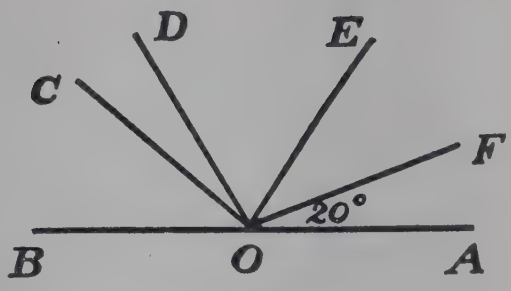


Fig. 2.

1. In fig. 1 ∠AOD is double each angle DOC and COB. What size is ∠DOC ?

2. In fig. 2 ∠FOA = 20°. What is the sum of angles FOE, EOD, DOC, COB ?

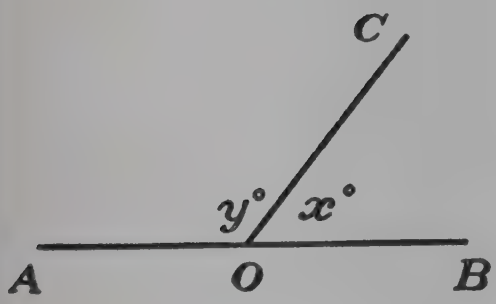


Fig. 3.

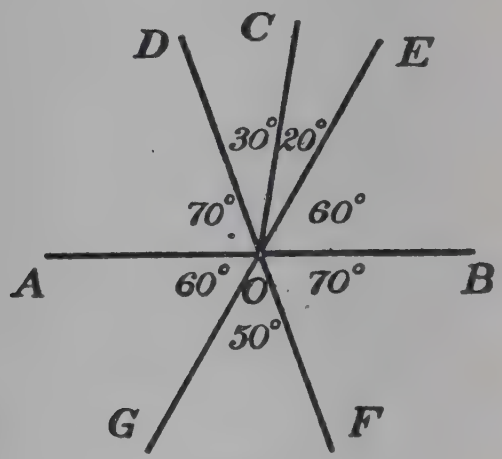


Fig. 4.

3. A straight line CO meets another straight line AB at O (fig. 3), making the adjacent angles COB and COA equal to $x°$ and $y°$ respectively. Find the values of x given that :—

(*a*) $5x = y$, (*b*) $x = \frac{2}{3}y$, (*c*) $y = 55$,
(*d*) $y = x + 50$, (*e*) $y = 5x - 20$.

4. Pick out any pairs of lines in fig. 4 which are in the same straight line.

5. Draw two adjacent angles whose sum is 180° (*i.e.* two adjacent supplementary angles).

6. How many degrees are there in the supplement of :—

(*a*) 47°, (*b*) 98°, (*c*) 174° ?

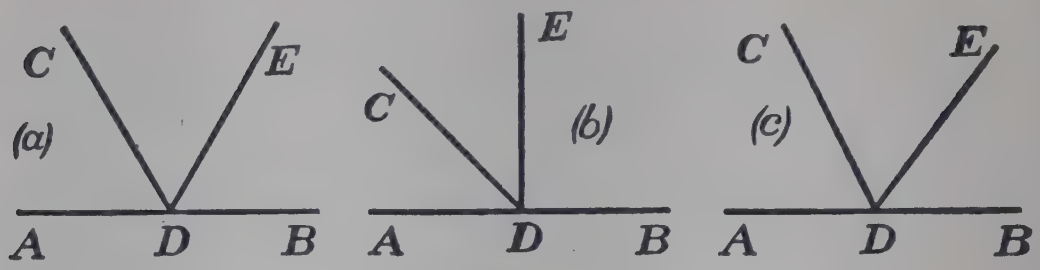


FIG. 7.

7. AB is a straight line, CD and ED meet it at D. Find in degrees the value of the angles BDE, EDC, CDA :—

(*a*) if they are equal;

(*b*) if ∠BDE is twice the size of each of the other angles;

(*c*) if ∠BDE is ten degrees less than each of the other angles.

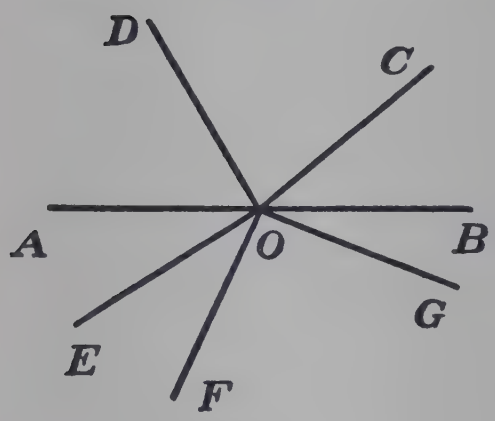


FIG. 8.

8. In fig. 8 AOB is a straight line.

(*a*) What is the sum of angles BOC, COD, and DOA ?

(*b*) What is the sum of angles AOE, EOF, FOG, GOB ?

(*c*) What is the sum of these seven successive angles at the point O ?

9. Six straight lines meet at a point O.

(*a*) What is the sum of the successive angles thus formed ?

(*b*) If these angles are equal, how many degrees will each contain ?

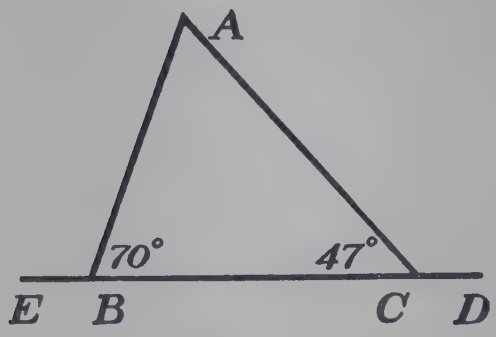


FIG. 10.

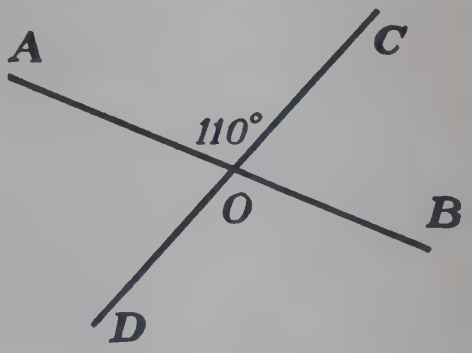


FIG. 11.

10. In fig. 10, how many degrees are there in :—

(*a*) ∠ACD, (*b*) ∠ABE ?

11. In fig. 11, what is the sum of ∠AOD, ∠DOB, and ∠BOC ?

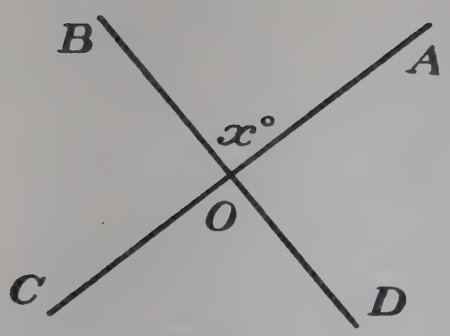


Fig. 12.

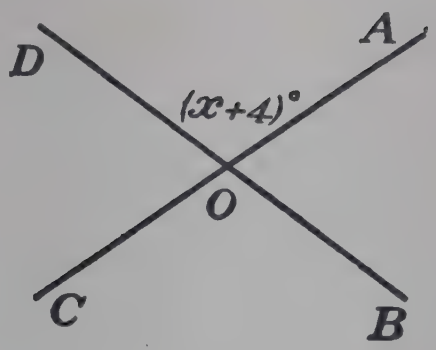


Fig. 13.

12. In fig. 12, what is the number of degrees in :—
(*a*) ∠AOD, (*b*) ∠DOC, (*c*) ∠COB ?

13. Two straight lines AC and DB cross at O, and ∠AOD contains $(x+4)°$. What is the sum of ∠AOB, ∠BOC, and ∠COD ?

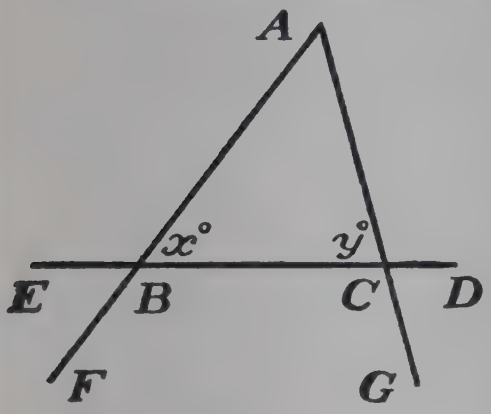


Fig. 14.

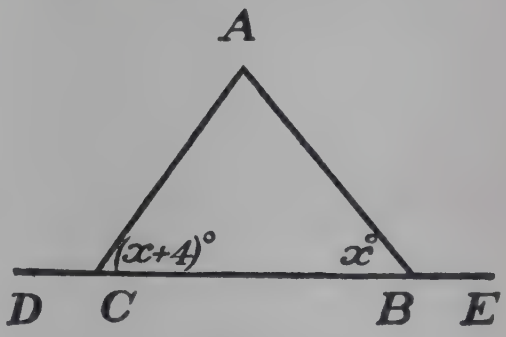


Fig. 15.

14. In fig. 14, how many degrees are there in each of the following angles :—
(*a*) ACD, (*b*) DCG, (*c*) ABE, (*d*) EBF ?

Middle.

15. In fig. 15, what is the magnitude of :—
(*a*) ∠ACD, (*b*) ∠ABE?

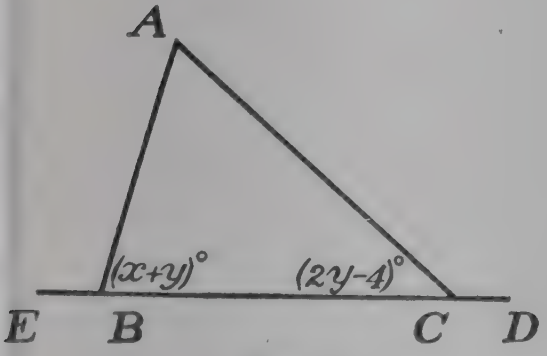


Fig. 16.

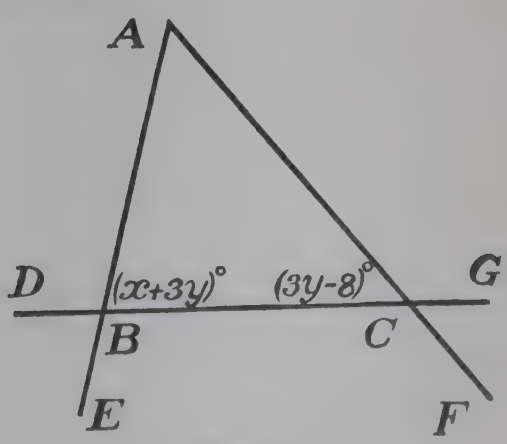


Fig. 17.

16. In fig. 16, what is the magnitude of :—
(*a*) ∠ACD, (*b*) ∠ABE ?

17. In fig. 17, what is the sum of :—
(*a*) Angles ABD, DBE and EBC ?
(*b*) Angles ACG, GCF and FCB ?

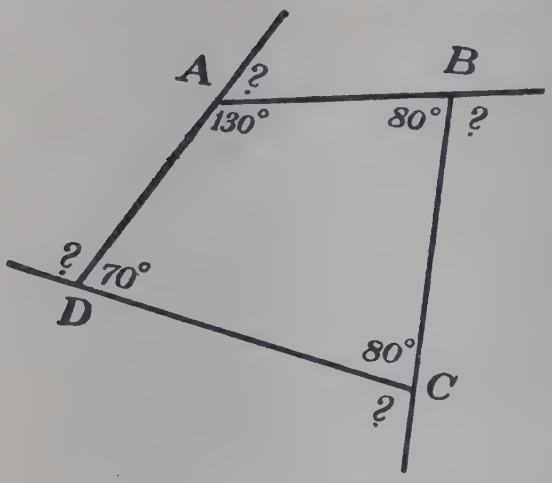


Fig. 18.

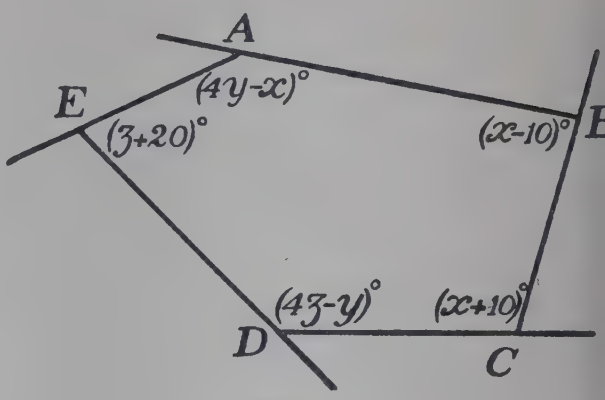


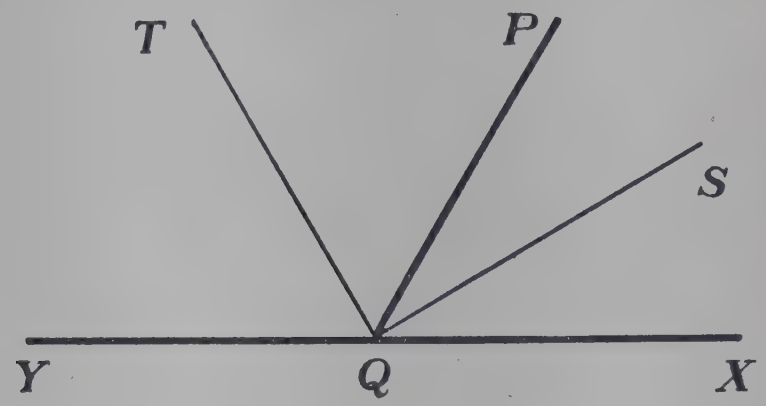
Fig. 19.

Higher.

18. How many degrees are there in each of the exterior angles of fig. 18 ? What is their sum ?

19. Find the number of degrees in each of the exterior angles of fig. 19. What is their sum ?

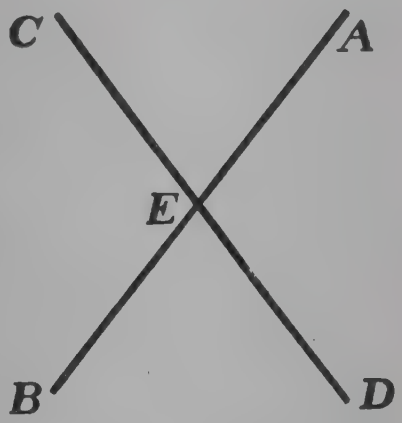
31. Example 1.— *A straight line* PQ *meets another straight line* YX *at* Q. *The adjacent angles* PQY *and* PQX *are bisected by the straight lines* TQ *and* SQ *respectively. Prove that the bisectors meet at right angles.*



To prove this :—

(*a*) Write down what you know about $\angle PQY + \angle PQX$.
(*b*) What follows about the value of $\frac{1}{2}\angle PQY + \frac{1}{2}\angle PQX$?
(*c*) What angle in the above figure equals $\frac{1}{2}\angle PQY + \frac{1}{2}\angle PQX$?
(*d*) What will be the value of this angle ?

EXAMPLE 2.—AB and CD are two straight lines crossing at E.



(*a*) What do we know about ∠AEC+∠CEB ?

(*b*) What do we know about ∠DEB+∠CEB ?

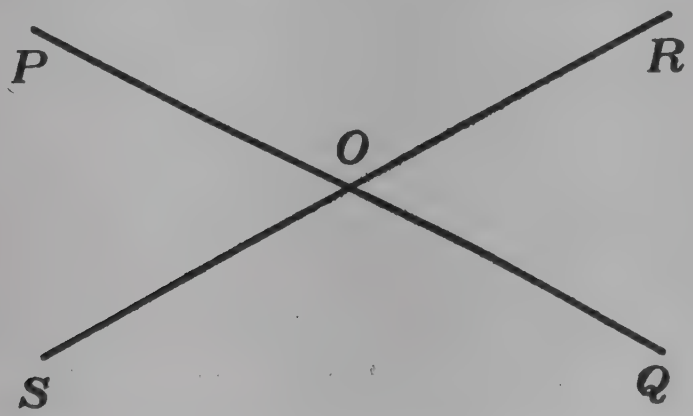
(*c*) From these two questions and answers, what can we tell about ∠AEC and ∠DEB ?

(*d*) What follows about ∠AED and ∠CEB?

32. Look through the last exercise again and notice that ∠CEB is used in both (*a*) and (*b*).

Draw two straight lines PQ *and* RS *cutting at* O.

Prove (1) ∠POS=∠ROQ, (2) ∠POR=∠SOQ.



We know that the point where the arms of an angle meet has a special name, **the vertex.**

The opposite angles made when two straight lines cut one another are called **vertically opposite angles.***

In the above figure ∠POR is vertically opposite ∠SOQ. What is their common vertex ?

What are the other two vertically opposite angles in the figure ?

Which are the pairs of vertically opposite angles made by a straight line AB crossing another straight line CD at X ?

What have we proved about vertically opposite angles ?

* They have the same vertex and their arms are in opposite directions.

33. Let us carefully examine the proof given in the last paragraph to see if we have followed the rules given in paragraphs 27 and 28.

In attempting to prove that vertically opposite angles are equal, we have not trusted to the fact that they **look** equal, or that, by **measurement** in the **particular** figure drawn, they can be shown to contain the same number of degrees, but we have based our conclusions on the **general fact** of Assumption 1.

Examples IIIb

Lower.

1. Draw any angle **POQ** and its vertically opposite angle **XOY**. What two angles are supplementary to ∠**POQ** ?

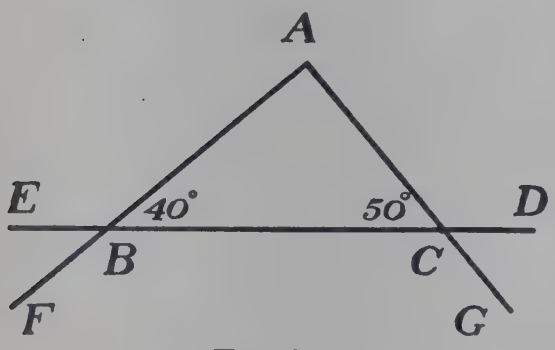


Fig. 1.

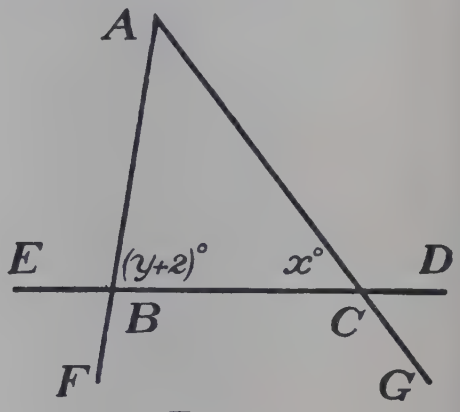
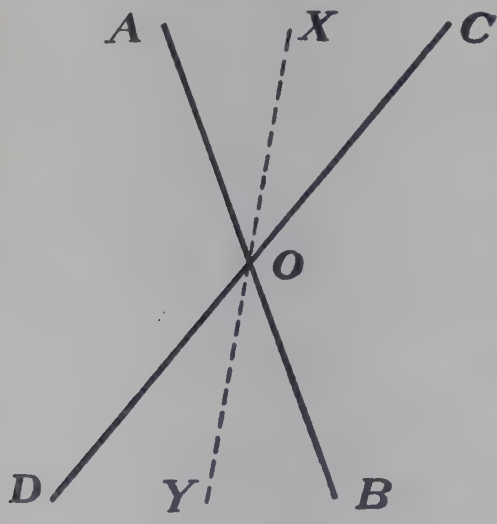


Fig. 2.

2. What is the magnitude of angles **ACD, DCG, GCB, CBF, FBE, EBA** in (*a*) fig. (1), (*b*) fig. (2) ?



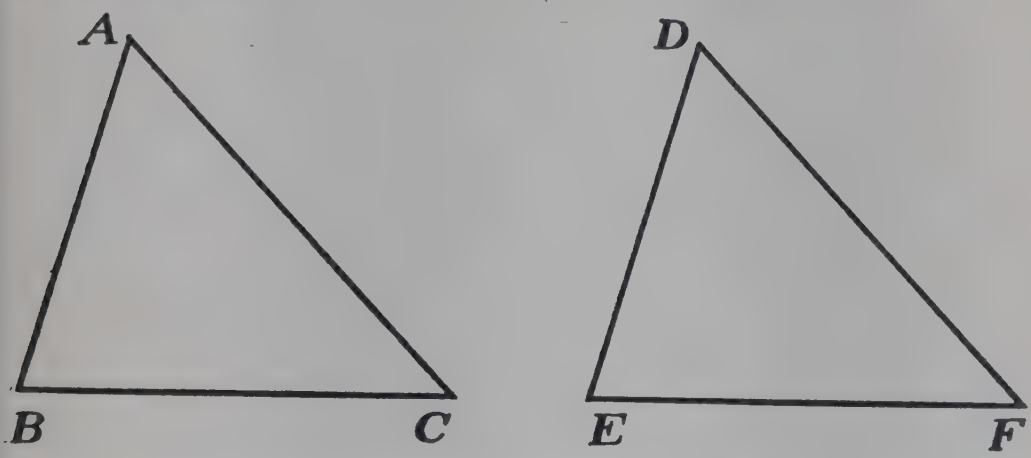
3. **AB** and **CD** are two straight lines cutting one another at **O**. **XO** and **YO** are straight lines drawn to bisect angles **AOC** and **DOB** respectively.

What do you know about :—

(*a*) angles **AOC** and **DOB** ?
(*b*) angles **XOC** and **BOY** ?
(*c*) ∠**AOX**+∠**XOC**+∠**COB**?
(*d*) ∠**YOB**+∠**XOC**+∠**COB**?
(*e*) ∠**XOC**+∠**COY** ?
(*f*) lines **XO** and **YO** ?

Give a reason for your answer in every case.

34. Read carefully **Assumptions 3 (a), (b), (c), and (d).** We must use one of these when proving that two triangles are equal in every respect. **If the two triangles can pass one of these tests, they are of the same shape and size,** *i.e.* they are **congruent.**



E.g. if the two triangles ABC, DEF have :—

$$* \begin{cases} (1)\ AB = DE \\ (2)\ AC = DF \\ (3)\ BC = EF, \end{cases}$$

they pass the test of 3(*a*). Notice carefully the bracket.

No one of these facts alone is sufficient to guarantee that the triangles are of the same shape and size. When, however, we can make **all three statements,** we may conclude that the triangles are equal in every respect and we then know that

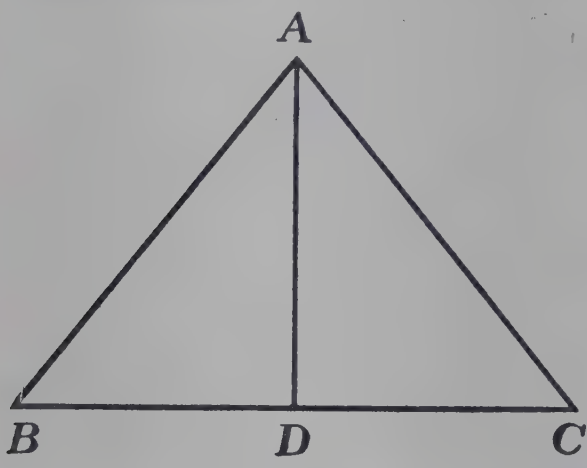
(1) $\angle ABC = \angle DEF$
(2) $\angle ACB = \angle DFE$
(3) $\angle BAC = \angle EDF$.

Notice carefully that $\angle ABC$ is equal to $\angle DEF$, *i.e.* it equals the angle **which corresponds to it** in $\triangle DEF$, and this rule holds good for the other angles of the triangles. So, also, in using Assumption 3 (*c*), we must be sure that the two sides given correspond to one another.

35. When we wish to prove two lines or two angles equal, it is often simplest to show that they are the corresponding parts of two congruent triangles.

* Another and briefer way of saying this is as follows: AB, AC, BC are respectively equal to DE, DF, and EF.

EXAMPLE 1.—*Triangle* ABC has AB=AC.* *Prove that* ∠ABD=∠ACD.



Draw the line AD bisecting ∠BAC and meeting line BC in D. The figure is then divided into two triangles which can be shown to be identically equal by Assumption 3 (*b*), and so the corresponding angles ABC, ACB are equal.

EXAMPLE 2.—*Using the assumption that two triangles are congruent if two angles and one side of the one are respectively equal to two angles and one side of the other, prove that* **a triangle which has two equal angles has two equal sides.**

Examples IIIc

Lower.

1. ABCD is a four-sided figure with AB=BC, AD=DC. Draw the straight line BD and prove that it divides ∠ABC into two equal parts.

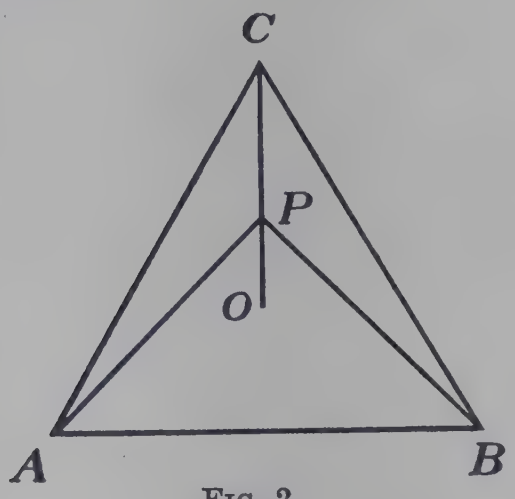


FIG. 2.

2. ABC is a △ with CB=CA. The straight line CO bisects ∠ACB; P is any point in CO. Prove that PA=PB.
(Consider △s CAP and CBP.)

* *Triangles with two equal sides are called* **isosceles** *triangles.*

3. Name any pairs of equal angles in fig. 2, giving a reason in every case for the equality of the angles.

4. **Prove that the perpendicular from the vertex of an isosceles triangle to the base:—**

(*a*) **bisects the vertical angle;**
(*b*) **bisects the base.**

5. **ABC** is an equilateral triangle (*i.e. all its sides are equal*). Prove that the bisector of any of its angles will, if produced to meet the opposite side, divide the triangle into two congruent triangles.

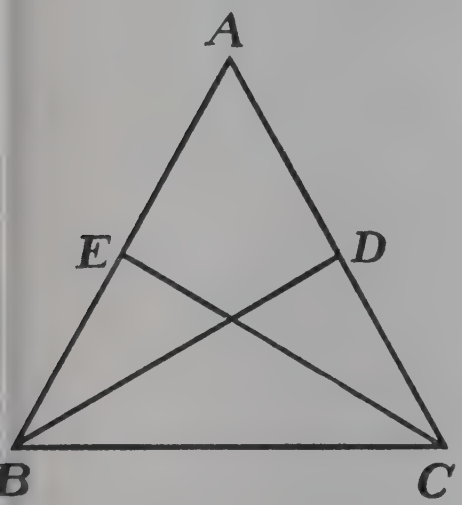


Fig. 6.

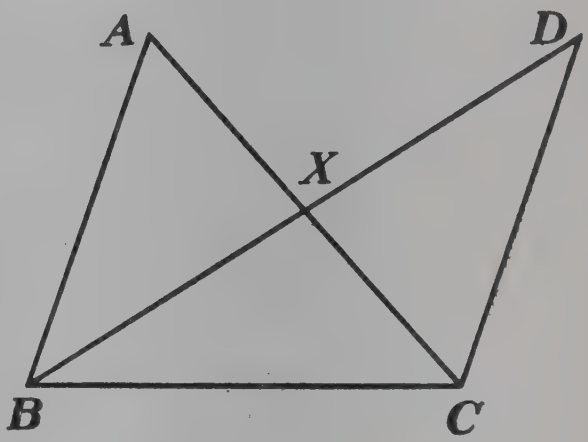


Fig. 7.

6. **ABC** is an isosceles triangle with **AB**=**AC**. **CE** is drawn to bisect **AB** at **E**, **BD** bisects **AC** at **D**.
Prove that △s **BEC** and **BDC** are congruent.

7. **ABC** is a △ with **X** the mid-point of **AC**. The straight line **BX** is produced to **D** so that **XD**=**BX**.
Prove that △s **ABX**, **DXC** are congruent.
What angle of △**DXC** equals ∠**XBA** ?

Middle.

8. **ABC** is an isosceles triangle with **AB**=**AC**. **CE** is drawn at right angles to **AB** at **E**, **BD** at right angles to **AC** at **D**.
Prove that △s **BEC** and **BDC** are congruent.

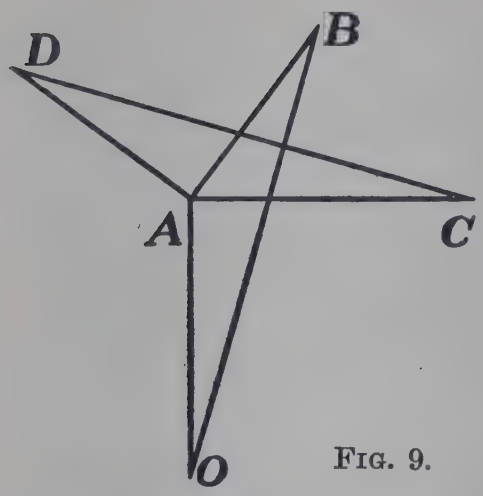
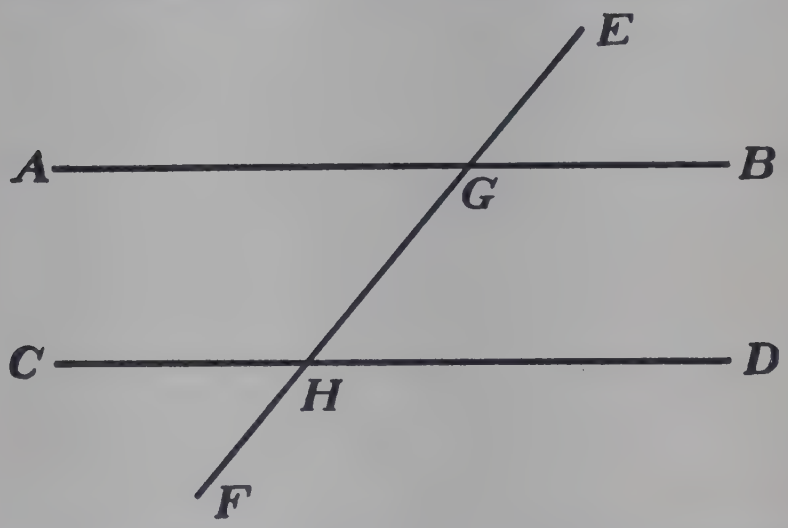


FIG. 9.

Higher.

9. **BAC** is a given angle. **AD** and **AO** are drawn at right angles to **AB** and **AC** respectively and **DA**=**BA**, **AO**=**AC**. Prove that **DC**=**BO**.

36. EXAMPLE 1.—We know that if a straight line crosses two parallel straight lines the corresponding angles are equal.



In the above figure, **AB** and **CD** are two parallel straight lines; **EF**, another straight line, crosses them at **G** and **H** respectively.

We know that $\angle EGB = \angle GHD$.

Since vertically opposite angles are equal

$$\angle EGB = \angle AGH,$$

and therefore $\angle GHD = \angle AGH$.

These inside angles on opposite sides of the cutting line are called **alternate angles.**

$\angle BGH$ and $\angle GHC$ are alternate angles; what can you prove about them?

EXAMPLE 2.—*In fig.* 2, **AB**, **CD**, **EF**, *and* **GH** *are parallel straight lines. Write down the names of as many pairs of alternate angles as possible.*

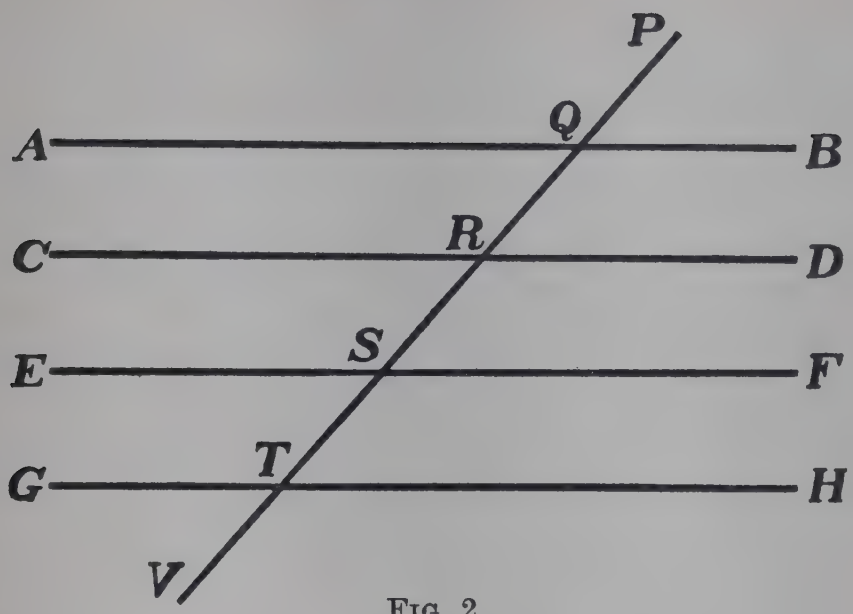


FIG. 2.

What have we proved about alternate angles ?
Never forget this.

EXAMPLE 3.—PQ and XY are two parallel straight lines ; another straight line AD crosses them at B and C respectively.

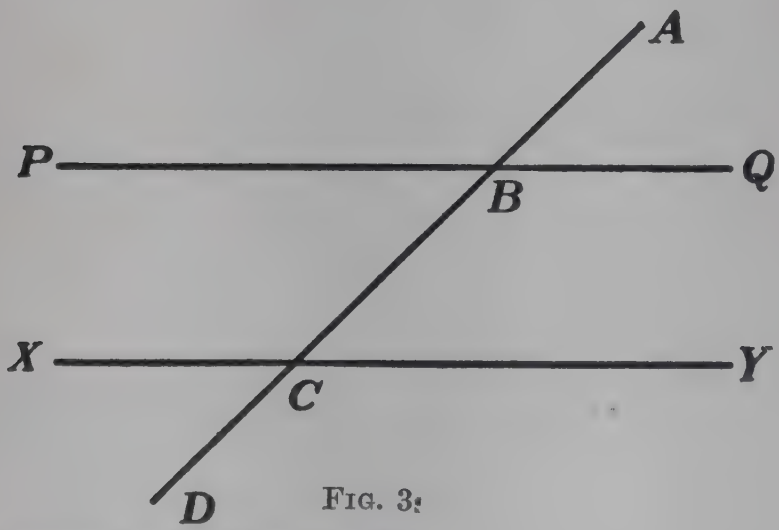


FIG. 3.

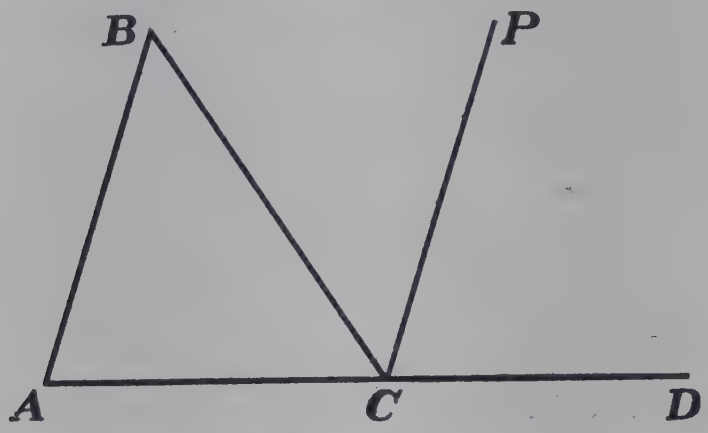
Consider angles QBC and BCY.
Let us write down what we already know about each of them.

(1) $\angle QBC + \angle ABQ = ?$ (How do we know this ?)
(2) $\angle BCY = \angle ABQ$. Why ?
(3) What would (1) become if we wrote $\angle BCY$ in place of $\angle ABQ$?

What then have we proved about the two inside angles on the same side of the cutting line ?

EXAMPLE 4.—*Prove the above fact for the two inside angles on the left-hand side of* AD *in the figure of Example* 3.

37. ABC is a △ with side AC produced to D. Through C a line CP is drawn parallel to AB.



AB and CP are two parallel straight lines and line ACD cuts them.

∴ (*a*) the corresponding angles are equal;

i.e. $\angle PCD = ?$

Also (*b*) the alternate angles are equal;

i.e. $\angle BCP = ?$

$\therefore \angle PCD + \angle BCP = ?$

To each side of this equation let us add $\angle BCA$.

Then $\angle PCD + \angle BCP + \angle BCA = ?$

But since ACD is a straight line and PC stands on it

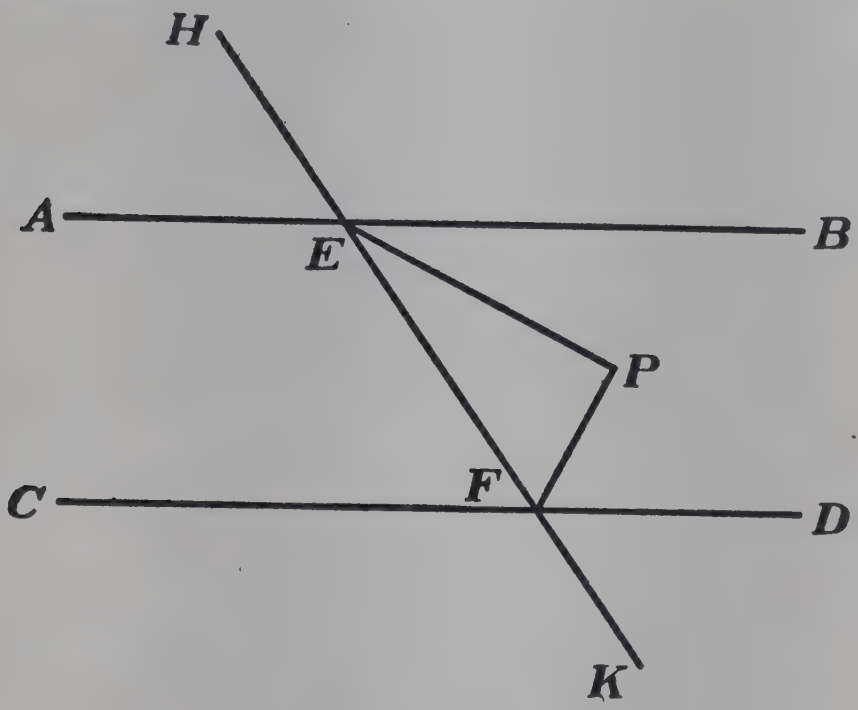
$\angle PCD + \angle BCP + \angle BCA = 180°,$

$\therefore \angle BAC + \angle ABC + \angle BCA = ?$

That is, we have proved that the sum of **the three interior angles of any triangle together equal two right angles.** You will remember that we have already used this fact, although we did not prove it, when we were constructing triangles.

38. In the course of the above proof, we have proved that, **if one side of a triangle be produced, the exterior angle is equal to the sum of the two interior opposite angles.** Go carefully through the proof till you come to this point.

39. AB and CD are two parallel straight lines crossed by a transversal HK at E and F respectively. ∠BEF and ∠EFD are bisected by lines which meet at P. Prove that ∠EPF is a right angle.



$$\angle BEF + \angle EFD = 180^\circ \text{ (why ?)}$$
$$\tfrac{1}{2}\angle BEF + \tfrac{1}{2}\angle EFD = 90^\circ$$
$$\therefore \quad \angle PEF + \angle EFP = 90^\circ$$
but $$\angle PEF + \angle EFP + \angle EPF = 180^\circ \text{ (why ?)}$$
$$\therefore \angle EPF = 90^\circ \text{ (*by subtraction*).}$$

40. If a straight line crosses two other straight lines and the alternate angles are equal, then the two straight lines are parallel.

This is called **a general statement,** because it is true not only for the particular pair of straight lines in a given figure but for all pairs of lines fulfilling the given conditions.

We call such general statements **theorems.**

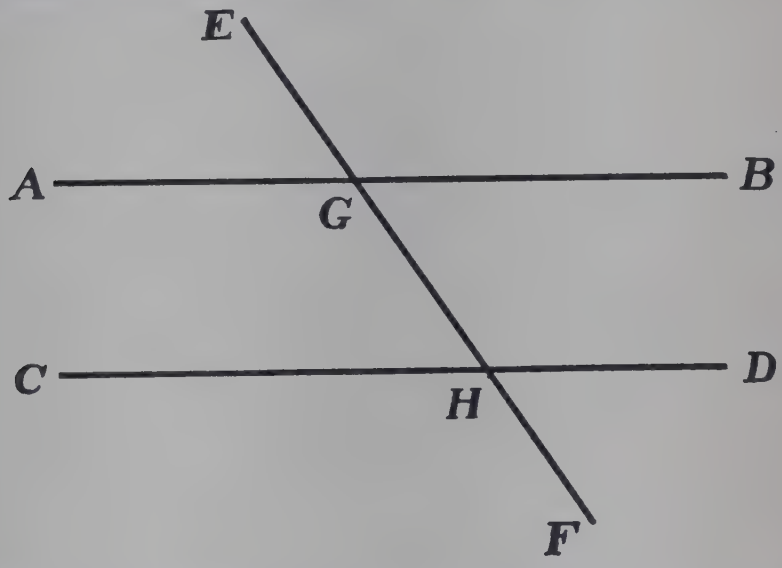
Many exercises in geometry are not theorems: they refer to some special figure which is indicated in the question. In such cases there is no general statement.

To prove the truth of a theorem, we draw *a figure* to which throughout the proof we may refer. We then write out the **general statement with reference to this figure,**

and, as we write it out, we divide it into two parts: (*a*) **what is given,** (*b*) **what is to be proved.** Next we state clearly what **construction** is necessary to help us with the proof, then we write **the proof** itself, and finally the **conclusion** at which we have arrived. The latter is the same in form as the general statement.

The next paragraph gives you an example.

41. *General Statement.*—**If a straight line crosses two other straight lines and the alternate angles are equal, then the two straight lines are parallel.**



Given.—A straight line EF crossing the straight lines AB and CD respectively and making $\angle AGH = \angle GHD$.

To Prove.—AB $\|$* CD.

Construction.—None.

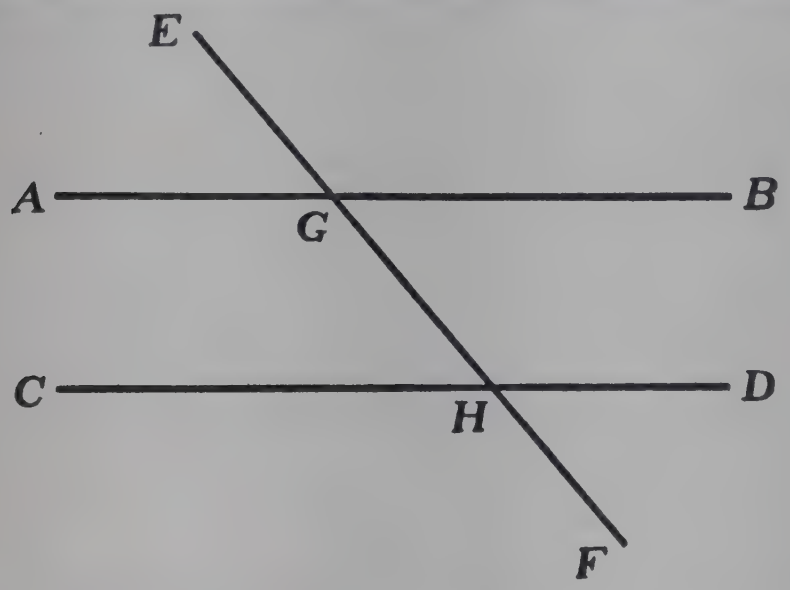
Proof :— $\angle AGH = \angle GHD$ (*given*).
$\angle AGH = \angle EGB$ (*vertically opposite*).
$\angle GHD = \angle EGB$ (*each equals* $\angle AGH$).
$\therefore$ AB $\|$ CD (*Assumption* 4 (*b*)).

Now the same reasoning can be applied when any straight line crosses two other straight lines and makes the alternate angles equal.

Conclusion.—**If a straight line crosses two other straight lines and the alternate angles are equal, then the two straight lines are parallel.**

* *The symbol* $\|$ *means* **is parallel to.**

42. *General Statement.*—**If a straight line crosses two other straight lines and makes the interior angles on one side of the transversal supplementary, then the two straight lines are parallel.**



Given.—A straight line **EF** crossing two other straight lines **AB** and **CD** at **G** and **H** respectively and making $\angle$**BGH** $+\angle$**GHD** $=180°$.

To Prove.—**AB** $\parallel$ **CD**.

Construction.—None.

Proof :—

$\angle$**BGH** $+\angle$**GHD** $=180°$ (*given*).

$\angle$**BGH** $+\angle$**BGE** $=180°$ (**EGH** *is a straight line*).

$\therefore \angle$**GHD** $=\angle$**BGE** (*supplementary to the same angle*).

$\therefore$ **AB** $\parallel$ **CD** (*Assumption* 4 (*b*)).

The same reasoning can be applied when any straight line crosses two other straight lines and makes the interior angles on one side of the transversal supplementary.

Conclusion.—**If a straight line crosses two other straight lines and the interior angles on the same side of the transversal are supplementary, then the two straight lines are parallel.**

43. If you wish your work to look well and your proof to be easy to follow, you must be very careful as to how you draw the figure.

(*a*) It should always be a reasonable size.

(*b*) It should be carefully drawn, but not necessarily to scale : this will depend on the question.

(*c*) You should take care that it fulfils all the conditions that you are given and not others; *e.g.* draw a triangle does not mean draw an isosceles triangle or a right-angled triangle.

(*d*) It should always be drawn so that anyone reading the proof can see the figure at the same time; this means that, if you turn a page, you must reproduce your figure.

Examples IIId

Lower.

1. In the figure of paragraph 37, what angles of △ **BAC** together equal ∠**BCD**? Produce **CB** to **E** and **BA** to **F**. What angles of the △ together equal: (*a*) ∠**ABE**, (*b*) ∠**CAF**?

2. **ABC** is a △ with **AB**=**AC**. **BA** is produced to **D** and **AG** is drawn parallel to **BC**. Name all the angles in the figure which equal ∠**ACB**.

3. **ABC** is an isosceles △ with **AB**=**AC**. **BD** and **CE** which cut at **O** bisect angles **ABC** and **ACB** respectively.

Prove that **BO**=**CO**.

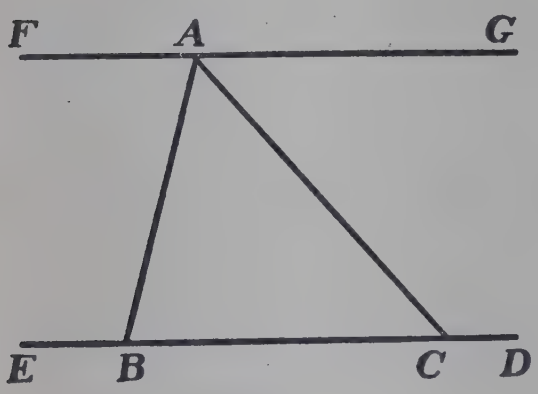


Fig. 4.

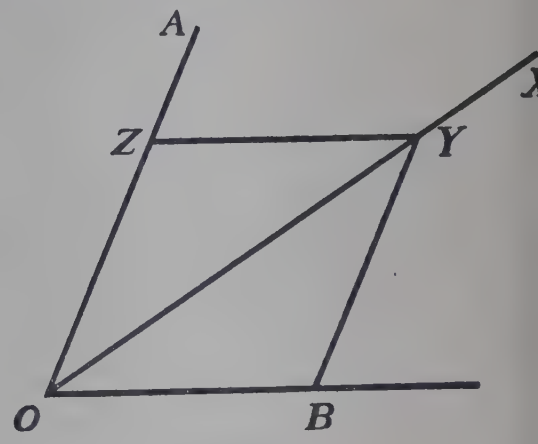


Fig. 5.

4. In fig. 4 the straight lines **FAG** and **EBCD** are parallel. Name any sets of three angles whose sum is 180°. Give a reason for your answer in every case.

5. ∠**AOB** is bisected by the straight line **OX**; **BY** is drawn parallel to **OA**, meeting **OX** in **Y**; **YZ** is drawn parallel to **BO**, meeting **OA** in **Z**.

Prove: (*a*) **OB**=**BY**; (*b*) **OZ**=**ZY**.

6. Prove that, in the figure of *Example* 5:—(*a*) △s **OZY** and **OBY** are congruent; (*b*) any two sides of figure **OZYB** are equal.

7. A point **D** is taken in the side **AB** of △ **ABC**. Through **D** a line is drawn parallel to **BC** and meeting **AC** in **E**. What do you know about △s **ADE** and **ABC**? What name do you give such triangles when you compare them?

8. **Two straight lines AB and CD are each parallel to a third straight line EF. Prove AB ‖ CD.**

9. AB and CD are two parallel straight lines with a transversal EF crossing them at E and F respectively; O is the mid-point of EF. Through O a second transversal is drawn cutting AB and CD at M and N respectively. Prove that △s MOE and FON are congruent.

10. AB and CD are two parallel straight lines with a transversal EF crossing them at E and F respectively. The straight lines ME and FN bisect angles AEF and EFD respectively. Prove: (*a*) ME ‖ FN; (*b*) ∠MEB = 180° − ∠NFD.

11. The vertical angle BAC of an isosceles △ is bisected by the straight line AD, which meets the base BC in D. Prove AD ⊥* BC.

12. Draw a straight line parallel to the base of an isosceles △. Prove that this line cuts the bisector of the vertical angle at right angles. (Produce the bisector of the vertical angle, if necessary.)

13. Prove that any straight line drawn parallel to the base of an isosceles △ makes equal angles with the other two sides or these sides produced.

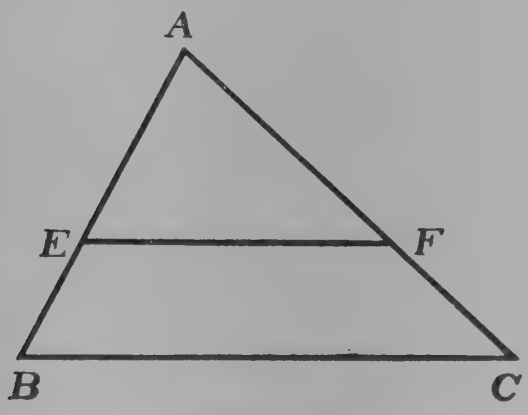


Fig. 14.

14. ABC is a △ with the straight line EF drawn parallel to the base BC and cutting AB and AC in E and F respectively. Name all the angles which are supplementary to ∠FEB, giving a reason for your answer in every case.

15. AB is a straight line, with two other straight lines AC and BD drawn perpendicular to it. Prove AB ‖ CD when :—

(*a*) they are on opposite sides of AB,
(*b*) they are on the same side of AB.

* *The symbol* ⊥ *means* **is perpendicular to** *or* **at right angles to.**

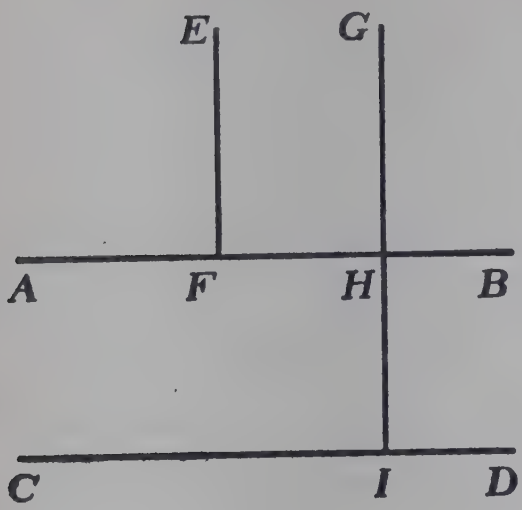


FIG. 16.

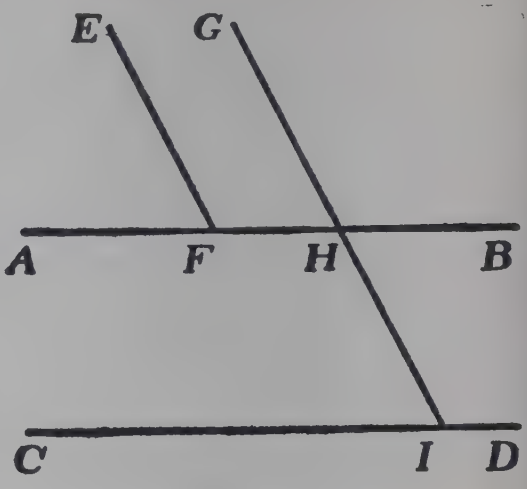


FIG. 17.

16. In fig. 16, AB and CD are two parallel straight lines. EF and GHI are drawn at right angles to AB and CD respectively.

Prove EF ‖ GI.

17. AB and CD are two parallel straight lines, and EF, another straight line, meets AB in F. A straight line GHI is drawn parallel to EF, and meeting AB and CD in H and I respectively.

Prove $\angle EFA = \angle HIC$.

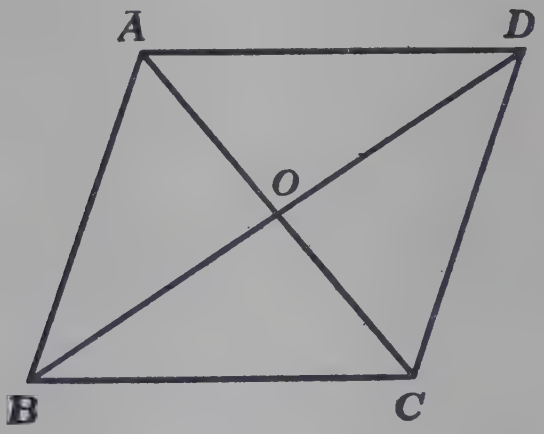


FIG. 18.

18. ABC is a $\triangle$ having O the mid-point of AC. The straight line BO is produced to D making OD = BO and lines AD, DC are drawn.

Prove: (*a*) AD ‖ BC; (*b*) CD ‖ AB.

Note carefully :—

(*a*) **A four-sided figure having its opposite sides parallel is a parallelogram.**

(*b*) **If a parallelogram has one of its angles a right angle, it is a rectangle.**

(*c*) **If a parallelogram has two adjacent sides equal, it is a rhombus.**

(*d*) **If a parallelogram has one of its angles a right angle and two adjacent sides equal, it is a square.**

19. Prove that the figure **OZYB** of *Example* 5 is a rhombus.

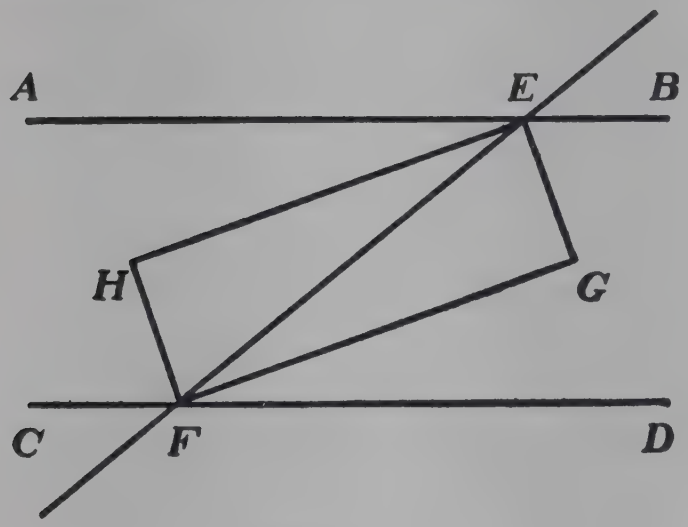


Fig. 20.

20. **AB** and **CD** are two parallel straight lines with a transversal crossing them at **E** and **F** respectively. The interior angles are bisected and the bisectors produced till they meet as in fig. 20.

Prove that the quadrilateral * formed by these bisectors is :—

(*a*) a parallelogram,
(*b*) a rectangle.

21. Reproduce the figure of *Example* 20, but let the transversal cross **AB** at right angles. Prove that the quadrilateral **HEGF** has not only : (*a*) its opposite sides parallel, (*b*) one of its angles a right angle, but also (*c*) two adjacent sides equal.

What name do we give to a quadrilateral which possesses all these qualities ?

* *A* **quadrilateral** *is bounded by four straight lines.*

22. **What is a parallelogram?** Draw a parallelogram and one of its diagonals.

Prove :—

(*a*) that **the diagonal divides the figure into two congruent triangles.**

(*b*) that **the opposite sides are equal.**

(*c*) that **the opposite angles are equal.**

23. Draw a parallelogram and its two diagonals.

Prove :—

(*a*) That the parallelogram is divided into two pairs of congruent triangles.

(*b*) That **the diagonals bisect one another.**

24. Draw a parallelogram **ABCD** and its diagonal **AC.**

(*a*) What is the sum of the interior angles of $\triangle$**ABC**?

(*b*) What is the sum of the interior angles of $\triangle$**ACD**?

(*c*) What is the sum of the four angles of the parallelogram?

25. **ABCD** is a parallelogram.

(*a*) What do you know about its opposite angles?

(*b*) What do you know about the sum of its four angles?

(*c*) What is the sum of angles **BAD** and **ADC**?

Give a reason for your answer.

26. **ABCD** is a parallelogram. Name four pairs of supplementary angles in the figure.

27. Prove that **any angle of a rectangle is a right angle.**

28. Prove that **any two sides of a rhombus are equal.**

29. Prove that **the diagonals of a rhombus :—**

(*a*) **bisect the angles through which they pass.**

(*b*) **cross at right angles.**

30. Prove that **the diagonals of a rectangle are equal.**

31. State all the facts of geometry you can prove about the diagonals of :—

(*a*) a parallelogram ;

(*b*) a rectangle ;

(*c*) a rhombus ;

(*d*) a square.

32. What is the sum of the interior angles of a quadrilateral ? [Draw one of its diagonals.]

33. **ABCD** is a quadrilateral with its opposite angles equal.

(*a*) What is the sum of its interior angles ?

(*b*) What is the sum of (i) angles **A** and **D** ? (ii) angles **A** and **B** ?

34. Prove that **a quadrilateral with its opposite angles equal is a parallelogram.**

35. Prove that **a quadrilateral is a parallelogram if:—**

(*a*) **its opposite sides are equal.**

(*b*) **its diagonals bisect one another.**

(*c*) **if it has one pair of sides which are both equal and parallel.**

36. Prove that **a parallelogram with equal diagonals is a rectangle.**

37. Prove that **a parallelogram with perpendicular diagonals is a rhombus.**

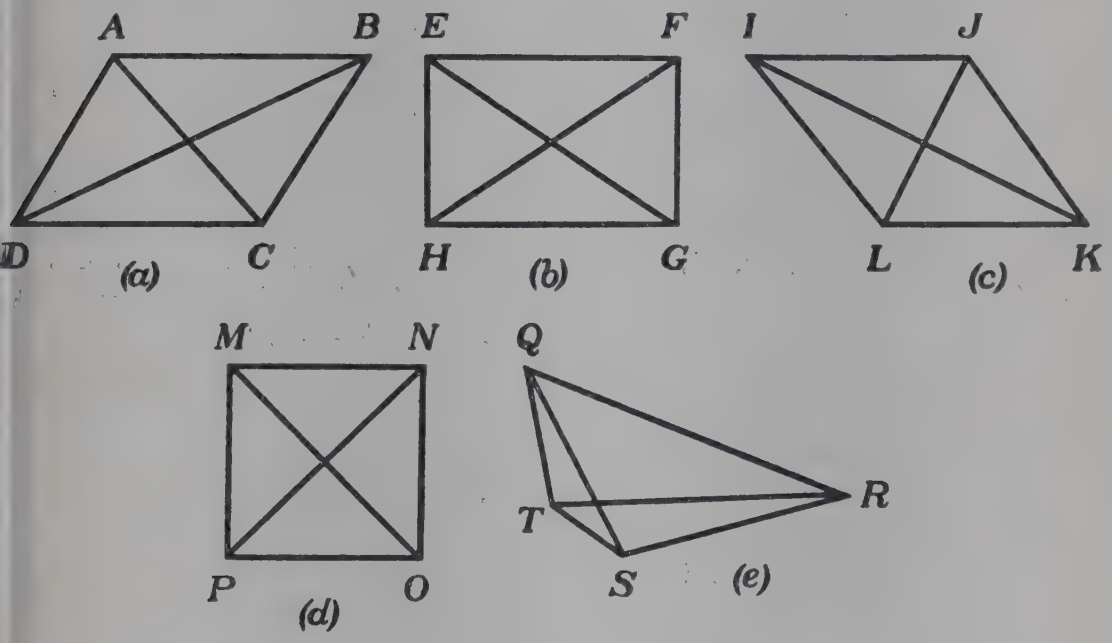


FIG. 38.

38. Of the five figures given above—

(*a*) has diagonals which bisect one another,

(*b*) has equal diagonals which bisect one another,

(*c*) has diagonals which bisect one another perpendicularly,

(*d*) has equal diagonals which bisect one another perpendicularly,

(*e*) has diagonals of which nothing is known.

By consideration of the diagonals alone, state which of these figures is:—

(*a*) a rhombus;

(*b*) a square;

(*c*) an irregular quadrilateral;

(*d*) a parallelogram;

(*e*) a rectangle.

39. **ABCD** is a parallelogram; the diagonal **BD** is drawn and it is found that $\angle$**DBC** $= 20°$ and $\angle$**CDB** $= 40°$. How many degrees are there in each of the remaining angles of the figure?

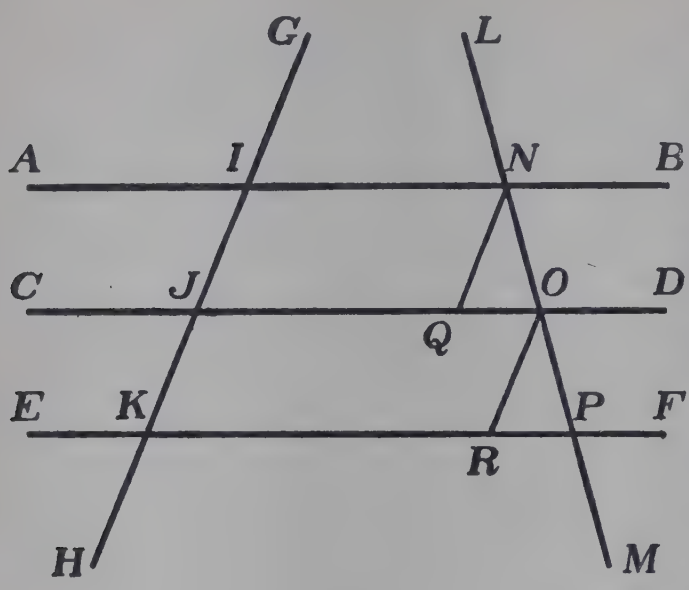


Fig. 40.

Middle.

40. AB, CD, and EF are three parallel straight lines which cut off equal intercepts IJ and JK from the transversal GH. NO and OP are the intercepts cut off from a second transversal LM. Prove that NO = OP.

[Draw NQ and OR parallel to GH as in fig. 40. Then prove :— (*a*) NQ = IJ = JK, (*b*) OR = JK = NQ, (*c*) △s NOQ and ROP to be congruent.]

41. D is the mid-point of the side BC of △ ABC. Through D a straight line is drawn parallel to AC and cutting AB in E. Prove that E is the mid-point of BA.

[Draw EF parallel to DC to meet CA in F and prove the congruency of △s BDE and EFA.]

42. ABC is a △ with D and E the mid-points of AB and AC respectively. Prove: (*a*) DE ∥ BC ; (*b*) $DE = \frac{1}{2}BC$.

[Produce DE to F making EF = DE. Draw the straight line FC and prove that DFCB is a parallelogram.]

43. Prove that the straight lines joining the mid-points of adjacent sides of a quadrilateral form a parallelogram.

[Draw one diagonal and use the facts proved in Nos. 42 and 35 (*c*).]

44. A quadrilateral has one pair of opposite sides equal and the other pair parallel. Is the figure necessarily a parallelogram ?

Illustrate your answer by a figure.

Higher.

45. Mentioning only essential properties, give definitions of :—

(*a*) a parallelogram ;
(*b*) a rhombus ;
(*c*) a rectangle ;
(*d*) a square.

Examples IIIe (*revision*)

Lower.

1. How many (*a*) right angles, (*b*) obtuse angles may a triangle have ?

2. How many angles of a triangle must be acute ? Give a reason for your answer.

3. Draw : (*a*) a right-angled isosceles triangle ; (*b*) an obtuse-angled isosceles triangle.

4. Name any kind of triangle which can have neither a right angle nor an obtuse angle.

5. An isosceles triangle has its vertical angle half the magnitude of each of the base angles. How many degrees are there in the vertical angle ? How many degrees would the vertical angle have contained if each base angle were one-sixth of the vertical angle ?

6. Triangle **ABC** has $\angle$**ABC** $= 2\ \angle$**BAC** and $\angle$**ACB** $= \frac{1}{2}\ \angle$**BAC**. Find the number of degrees in each angle.

7. Triangle **ABC** has $\angle$**ABC** $+\ \angle$**ACB** $= 91°$ and $\angle$**ABC** $-\ \angle$**ACB** $= 42°$. Find the magnitude of each of these angles.

8. Two angles of a $\triangle$ together contain $(x+2)°$; their difference is $(y+1)°$. Find the number of degrees in each angle.

9. A triangle **ABC** has its side **BC** produced to **D** ; $\angle$**BAC** $= 60°$, $\angle$**ABC** $= 50°$. What is the magnitude of : (*a*) $\angle$**ACB**, (*b*) $\angle$**ACD** ?

10. A triangle **ABC** has its side **BC** produced to **D** and side **CB** to **E** ; $\angle$**ACD** $= 130°$, $\angle$**ABE** $= 110°$. Find in degrees the value of : (*a*) $\angle$**BCA**, (*b*) $\angle$**CBA**, (*c*) $\angle$**BAC**.

11. The side **BC** of $\triangle$ **ABC** is produced to **D** and **CB** to **E** ; $\angle$**ACD** $= (x+4)°$, $\angle$**ABE** $= (y-1)°$. Find the number of degrees in : (*a*) $\angle$**BCA**, (*b*) $\angle$**CBA**, (*c*) $\angle$**BAC**.

12. One angle of a triangle is double the second angle ; the third angle is 5° greater than the first. Find in degrees the magnitude of each angle.

13. An isosceles triangle has its vertical angle one-third of the sum of the other two angles. What is the number of degrees in each of the angles opposite the equal sides ?

14. Can you draw a triangle having the value of its angles as follows : (*a*) $\angle$**A** $= 60°$, (*b*) $\angle$**B** $= 50°$, (*c*) $\angle$**C** $= 80°$? If this be impossible, keeping angles **A** and **B** as given, what alteration in **C** would be necessary ?

Similarly, find the necessary alteration for the other angles in turn.

15. What fraction of a complete revolution is an angle of an equilateral triangle ?

16. **ABC** is a $\triangle$; **BC** is produced both ways. Prove that the sum of the exterior angles thus formed is more than 180°.

17. In $\triangle$ **ABC**, **BO** and **CO** are drawn to bisect angles **ABC** and **ACB** respectively and meet at **O**. Prove that $\angle$**BOC** is more than a right angle.

18. Prove that an isosceles triangle has not less than two acute angles.

19. Angle **B** of a $\triangle$ **ABC** is a right angle ; $\angle$**ACB** is two-thirds of $\angle$**BAC**. Find the number of degrees in : (*a*) $\angle$**BAC**, (*b*) $\angle$**ACB**.

20. One side of a triangle is produced, and the bisector of the exterior angle thus formed is found to be parallel to one of the sides of the $\triangle$. Prove that the $\triangle$ is isosceles.

21. Prove that the line drawn from the vertex of an isosceles triangle to the mid-point of the base divides the figure into two congruent triangles

22. An isosceles triangle has its vertical angle **BAC**$=54°$, and the base **BC**$=3\frac{1}{2}$ ins. **AD** is drawn at right angles to **BC** and meeting it in **D**. What is (*a*) the length of **BD**; (*b*) the number of degrees in $\angle$**BAD** ?

23. What is the angle sum of (*a*) a triangle; (*b*) a quadrilateral ?

24. Draw a five-sided figure with each of its angles less than 180°. What is the sum of its interior angles ? Find the sum of the interior angles when the figure has (*a*) six sides, (*b*) seven sides.

25. What is the number of degrees in the interior angle of a regular * figure which has (*a*) five sides; (*b*) six sides; (*c*) eight sides ?

26. Prove that the diagonals of a regular five-sided figure are equal.

27. Draw any five-sided figure with each of its angles less than 180°. Produce the sides in order. [See fig. 27.]

(*a*) What is the sum of any interior angle with its adjacent exterior angle ?
(*b*) How many vertices has the figure ?
(*c*) What is the sum of all the interior and exterior angles ?
(*d*) **What is the sum of all the interior angles ?**
(*e*) **What is the sum of all the exterior angles ?**

* *A* **regular** *figure has all its sides equal and all its angles equal.*

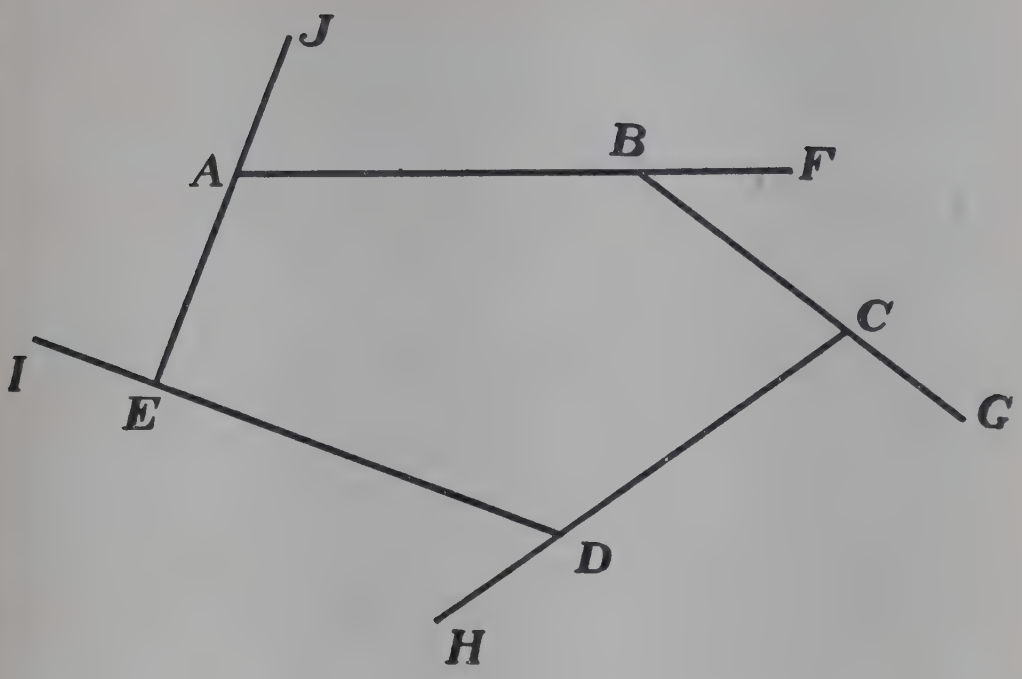


FIG. 27.

28. Repeat Exercise 27 with (*a*) a six-sided figure; (*b*) an eight-sided figure; (*c*) a nine-sided figure.

29. Look at your answers to Exercise 28; **what do you notice about the sum of the exterior angles in each case**? Would this be true if your figure had: (*a*) three sides; (*b*) four sides; (*c*) ten sides?

30. Suppose you drew figures as in Question 27, but with (*a*) p sides; (*b*) n sides; (*c*) $2r$ sides, what would be the sum of:—

(i) **the interior angles,**
(ii) the interior and the exterior angles,
(iii) **the exterior angles**

in each case?

31. How many degrees are there in the exterior angle of a regular nine-sided figure? What size is its interior angle? Construct the figure with a side of 2 ins.

32. How many sides has a regular polygon * if its exterior angle contains: (*a*) 24°, (*b*) 30°, (*c*) 60°, (*d*) 90°?

33. How many sides has a regular polygon if its interior angle is: (*a*) 90°, (*b*) 108°, (*c*) 156°?

* *A* **polygon** *is bounded by more than four straight lines.*

34. What is the number of degrees in the remaining angle of a five-sided figure which has four of its angles respectively: (*a*) 140°, (*b*) 76°, (*c*) 38°, (*d*) 120° ?

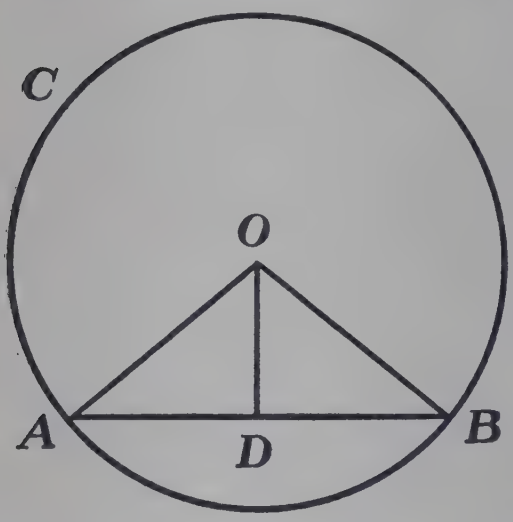
35. **Enumerate the different tests you could apply to a quadrilateral to see if it were a parallelogram.**

Middle.

36. Find the remaining angle of a five-sided figure which has four of its angles respectively: $x°$, $(x+4)°$, $(3y+2)°$, $(2y+1)°$.

37. What regular figure has an interior angle which is $\frac{14}{15}$ of the interior angle of a regular twelve-sided figure ?

44. *General Statement.*—**If a straight line from the centre of a circle bisect a chord,* it is perpendicular to the chord.**



Given.—D the mid-point of chord AB in ⊙ † ABC and O the centre of the ⊙.

To be proved.—The straight line OD ⊥ AB.

Construction.—Draw the straight lines OA, OB.

Proof.—Consider △s AOD and DOB.

1. OA = OB (*radii of same* ⊙).
2. AD = DB (*given*).
3. OD is common to both.

∴ the △s are congruent (*Assumption* 3 (*a*)).

∴ ∠ODA = ∠ODB (*corresponding angles of congruent* △s), but ∠ODA + ∠ODB = 180° (ADB *is a straight line*).

∴ each of these angles is a right angle;

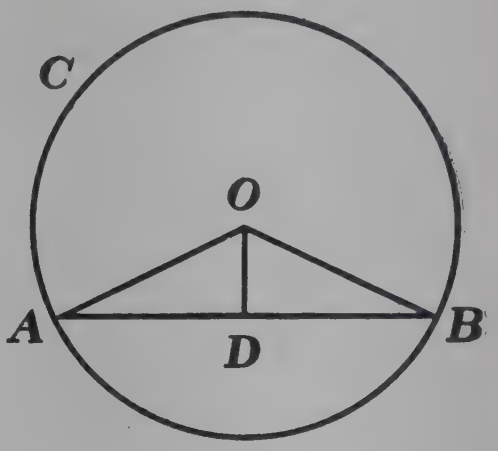
i.e. OD ⊥ AB.

Conclusion.—**If a straight line from the centre of a circle bisect a chord, it is perpendicular to the chord.** (*Compare this theorem with Exercise* 21, *Examples III e.*)

* *A* **chord** *of a circle is a straight line joining two points on the circumference.*

† ⊙ *is the symbol for circle.*

45. *General Statement.*—**If a straight line from the centre of a circle be perpendicular to a chord, it bisects the chord.**



Given.—A circle ABC with centre O, chord AB, and straight line OD perpendicular to AB and meeting it in D.

To prove AD=DB.

Construction.—Draw the straight lines OA, OB.

Proof.—Do this yourselves, using Assumption III (*d*).

46. Notice in your proof of the congruency of the △s ODA and ODB in the figure of paragraph 45 that :—

(*a*) OD bisects the base AB of the isosceles △DAB.

(*b*) OD bisects the vertical angle of △OAD, and remember that OD ⊥ base AB.

We call the line OD **a line of symmetry** since it divides △OAD into two parts such that, for every point in the one part, there is a corresponding point in the other part.

It is as if OD were a mirror in which every point in the △OAD could look and see its companion point behind the mirror, in the △OBD.

Do you know any other lines of symmetry ?

47. The most completely symmetrical figure that you can draw on your paper is a circle. It is symmetrical about every straight line of indefinite length drawn through its centre to cut the circumference in two points.

The part of such a line cut off and enclosed by the circumference is called a **diameter**: the diameter is twice the length of the radius. All diameters of a circle are equal.

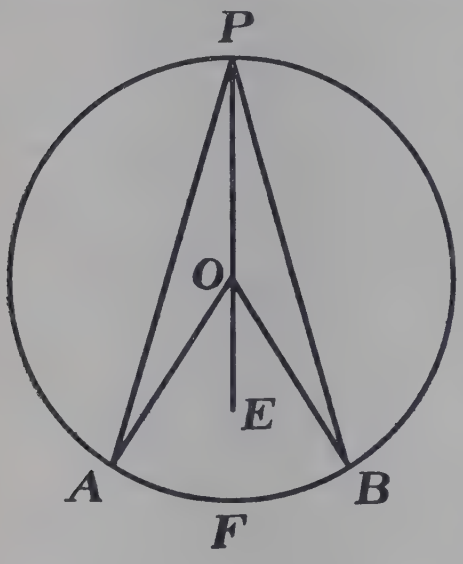
Do not forget that **the circle is the whole figure** ; when we wish to speak specially of **the line bounding it**, we refer to it as **the circumference.**

Half the circumference is **a semi-circumference** ; half the circle is **a semicircle.**

The curved line between any two points on a circumference is called **an arc.**

A segment of a circle is the part of a circle bounded by a chord and the part of the circumference which the chord cuts off.

Draw a number of circles and fill in appropriately the names of the above-mentioned parts.



48. *General Statement.*—**The angle at the centre of a circle is double an angle at the circumference standing on the same arc.**

Given.—O the centre of circle APB, ∠AOB at the centre, and ∠APB at the circumference stand on the same arc AFB.

To prove ∠AOB=2∠APB.

Construction.—Draw the straight line PO and produce it to any point E.

Proof :—

OP=OA	(*radii of a* ⊙).
∴ ∠APO=∠PAO	(*base angles of an isosceles* △).
∴ 2∠APO=∠APO+∠PAO	(*by addition*),
but ∠AOE=∠APO+∠PAO	(*the exterior angle of a* △ *equals the sum of the two interior opposite angles*).
∴ ∠AOE=2∠APO	(*both equal to* ∠APO+∠PAO).

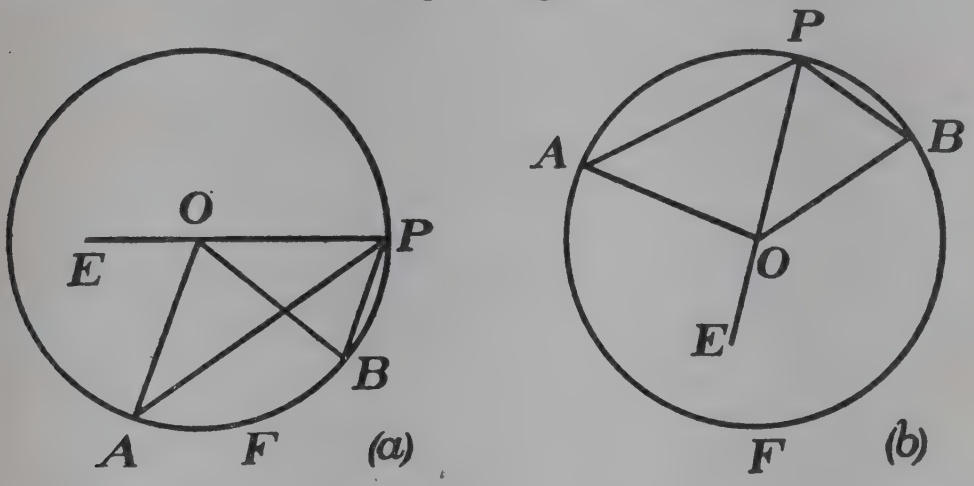
In the same way it can be shown that ∠EOB, the exterior angle of △OPB=2 ∠OPB.

Hence ∠AOE+∠EOB=2 ∠APO+2 ∠OPB (*by addition*), which gives ∠AOB=2 ∠APB.

Conclusion.—**The angle at the centre of a circle is double an angle at the circumference standing on the same arc.**

49. If you describe other circles each with an angle at the centre and an angle at the circumference standing in the same arc, you will find that the figures are not all as simple as the one in paragraph 48.

You may find that you get a figure similar to (*a*) or (*b*).



A proof on the same lines will hold, only, in these cases, it is a little more difficult to follow from the figures.

Those who attempt the Higher, or even the Middle, Syllabus should try to prove the theorem from these figures.

50. In every exercise in geometry we must consider—

(*a*) **which of the facts at our disposal we shall use ;**

(*b*) **the way in which we shall use these facts.**

You will have noticed by this time that the **theorems** proved give you an ever-increasing amount of material which, with your **assumptions** and the **facts given** in a particular problem, supply you with the help that is necessary for the solution of that problem.

We must, then, search for suggestions as to the assumptions or previous theorems which are likely to help us in arriving at the proof. It is well worth while, after drawing the figure and before beginning the proof, to go over in our minds with reference to the figure each of the facts given and each of the facts to be proved, so that we may decide to what section of the work the problem appears to belong.

If we are to prove that two lines are equal, it may be necessary to show that they are the corresponding sides of

two congruent triangles, or the opposite sides of a parallelogram, or the diagonals of a rectangle, etc., etc. : if we have to prove that two angles are equal, it may be necessary to show that they are the corresponding angles of two congruent triangles, or vertically opposite angles, or the opposite angles of a parallelogram, etc., etc.

Examples IIIf

Lower.

1. **If a four-sided figure is inscribed * in a circle, its opposite angles are together equal to two right angles.**

Write this out and prove it in proper theorem form.

The following are two facts you will need in your proof :—

(*a*) The angle at the centre of a ⊙ is double an angle at the circumference standing on the same arc.

(*b*) The successive angles at a point are together equal to 360°.

2. **Prove that the angle in a semicircle is a right angle.**

3. **Prove that angles in the same segment of a circle are equal.**

4. Two chords **AB** and **CD** of a circle **ABCD** intersect at **O.** Prove that △s **OAD, OCB** are equiangular.

5. Prove that a parallelogram inscribed in a circle is a rectangle.

6. Prove that if the side **CD** of a cyclic † quadrilateral **ABCD** be produced to **E,** ∠**ADE**=∠**ABC.**

7. **AB** and **CD** are two equal chords of a circle **ABCD** ; from the centre **O,** perpendiculars **OF, OE** are drawn to **AB, CD** respectively. Prove **OF**=**OE.**

8. Prove that **if, from the centre of a circle, a perpendicular is drawn to each of two chords, and these perpendiculars are equal** (*i.e. he chords are equally distant from the centre*), **then the chords are equal.**

9. **ABCDEF** is a hexagon inscribed in a circle and the diagonal **AD** is drawn. Use the theorem proved in Exercise 1 of these examples to find the angle sum of the hexagon.

10. **BA** and **DC,** two chords of a circle, are produced to meet at **P.** Lines **AD, BC** are drawn.

Prove ∠**BCP**=∠**DAP.**

11. Refer to *Example* 10. If **AP**=**CP,** prove that **BC** and **DA** are equally removed from the centre of the circle.

* *A figure is* **inscribed** *in a circle when all its corners or vertices lie on the circumference of the circle.*

† *A quadrilateral inscribed in a circle.*

12. **AB** and **AC** are equal chords of a ⊙; ∠**BAC**=40°. **P** is a point on the circumference but not on the arc **ABC**. Find the magnitude of ∠**APC**.

13. **BC** is a diameter of a ⊙ with **A**, a point on the circumference. A perpendicular is drawn from **A** to meet **BC** in **D**.

Prove that △s **BAC** and **ABD** are equiangular.

14. **AD**, a diameter of a ⊙, bisects a chord **BC** at **E**. Prove that △s **ABE**, **ADC** are equiangular.

15. **ABC** is a △ inscribed in a ⊙. **AD** bisects ∠**BAC** and meets the circumference again in **D**.

Prove that △ **BDC** is isosceles.

Middle.

16. **ABCD** is a cyclic quadrilateral with **AD** less than **BC**, **AE** is drawn parallel to **DC**, meeting **BC** in **E**. Prove ∠**ABE**=∠**EAD**.

17. Prove that the difference between the angles at the extremities of one side of a cyclic quadrilateral is equal to the angle between that side and the opposite side when they are produced to meet.

51. *When you are told to construct a geometrical figure :—*

(1) Draw at the side of your paper a rough sketch of what the figure will be like when the construction is done.

(2) Study this sketch to find out:—

(*a*) the lines about which you are told sufficient to be able to draw them accurately ;

(*b*) the lines for which you have to find an accurate construction ;

(*c*) if there are any lines you **can** draw which will help you to obtain the lines mentioned in (*b*).

(3) Draw lines (*a*) and (*c*) accurately, leaving in all arcs, etc., used in the construction. You should then be able to draw lines (*b*).

(4) Study your figure to see on what theorems or assumptions the construction is based.

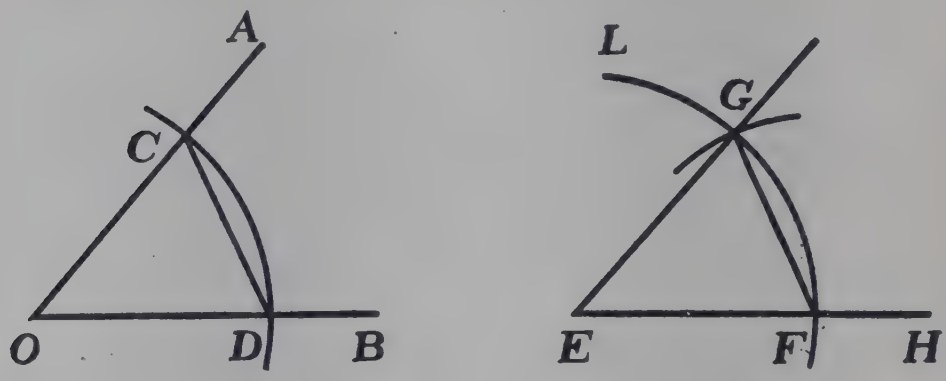
(5) Prove that your construction is correct, *i.e.* that the lines (*b*) are the lines for which you were asked.

(6) Verify your construction by some measurement if possible.

In the construction work you did last year you were not asked for (4) and (5) : many of the following examples are familiar to you, but they must now be done in complete mathematical style, *i.e.* **you must prove your construction to be correct.**

Remember that graduated rulers and protractors should be used as little as possible. In Examples IIIg, you will be told when protractors may be used.

52. Construct an angle equal to a given angle.



Given.—An angle AOB.

To be constructed.—An angle equal to angle AOB.

Construction :—

1. Draw the straight line EH.
2. With centre O and any radius, describe an arc cutting OA in C and OB in D.
3. With centre E and the same radius, describe an arc FL, cutting EH at F.
4. With centre F and radius the length of the chord DC, describe an arc cutting the arc FL at G.
5. Draw the straight lines EG, GF, CD.

Then $\angle GEF = \angle AOB$.

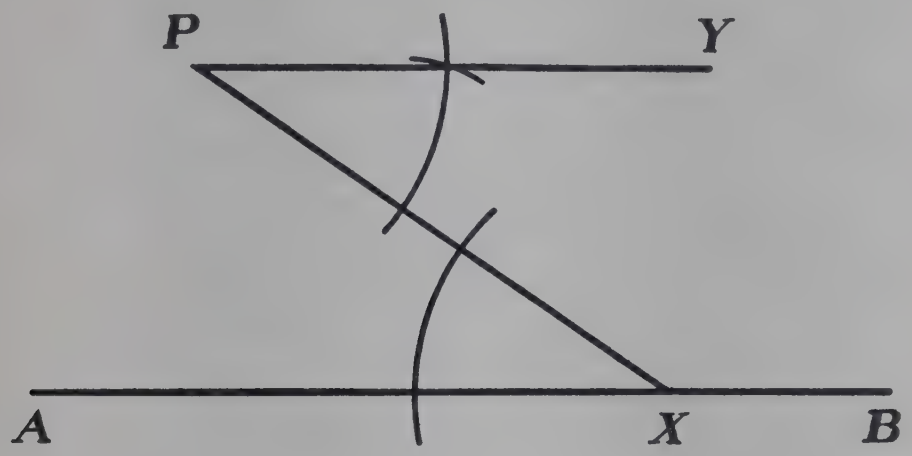
Proof:—Consider $\triangle$s COD and GEF.

$$\begin{cases} 1.\ OD = EF & (\textit{radii of equal circles}). \\ 2.\ OC = EG & (\quad ,, \quad ,, \quad ,, \quad). \\ 3.\ DC = FG & (\textit{by construction}). \end{cases}$$

$\therefore$ the two triangles are congruent (*Assumption IIIa*).

$\therefore$ $\angle GEF = \angle COD$ (*corresponding $\angle$s of congruent $\triangle$s*).

53. Through a given external point to draw a straight line parallel to a given straight line.



Given.—P is a point outside a straight line AB.

To be constructed.—A line through P parallel to AB.

Construction :—

1. Draw any straight line PX meeting AB at X.
2. At P construct $\angle XPY = \angle PXA$, but on the opposite side of PX.

Then PY, drawn through P, is parallel to AB.

Proof :—

$\angle XPY = \angle PXA$ (*by construction*).

$\therefore$ PY $\parallel$ AB (*since the alternate angles* XPY *and* PXA *are equal*).

Examples IIIg

Lower.

1. Bisect a given straight line.
2. Bisect a given angle.
3. Construct an angle equal to a given angle.
4. Construct a triangle, given the lengths of its three sides.
5. Construct a triangle, given two angles and the side they subtend.
6. Construct a triangle, given two sides and the angle included by them.

7. Construct a triangle, given two sides and the angle opposite one of them.

(Remember there are two possible solutions of this problem.)

8. Draw a perpendicular to a given straight line from a point (*a*) on it, (*b*) outside it.

9. Construct a right-angled triangle, given the length of the hypotenuse and one other side.

10. Construct an equilateral triangle with a side 2 in. in length.

11. Construct an angle of 60°. (Use construction of No. 10.)

12. Construct an angle of (*a*) 30°, (*b*) 120°.

13. Construct an angle of $22\frac{1}{2}°$.

14. Construct a parallelogram **ABCD**, given **AB**=3 in., **BC**=2 in., $\angle$**ABC**=60°.

15. Construct a parallelogram **ABCD**, having **AB**=3 in., **BC**=2 in., **AC**=4 in.

16. Construct a rhombus with one diagonal $4\frac{1}{2}$ in. and the other 5 in. in length.

17. Construct a rectangle **ABCD** with **AB**=3 in. and **AC**=5 in.

18. Construct an isosceles triangle with its equal sides 2 in. and the vertical angle 45°.

19. Construct an isosceles triangle **ABC** with base **BC**=4 in. and **AD**, the perpendicular from **A** to the base, equal to $3\frac{1}{2}$ in.

20. Construct a regular pentagon* on a 2 in. side. (Protractors may be used.)

21. Construct a parallelogram **ABCD**, given **AB**=4 in., $\angle$**DAB**=80°, $\angle$**DBA**=50°. (Protractors may be used.)

22. Construct a square on a 2 in. side.

Middle.

23. Construct an isosceles triangle **ABC**, with base **BC**=4 in., $\angle$**BAC**=90°.

Higher.

24. Construct a square with a 3 in. diagonal.

25. Refer to *Examples IIId*, No. 40, and try to devise a method for dividing a straight line into (*a*) three, (*b*) four, (*c*) five equal parts.

Examples IIIh (*oral*)

1. How would you test to see if:—
   (*a*) two triangles were congruent?
   (*b*) two straight lines were parallel?

* A pentagon is a polygon of five sides.

2. What assumption or what theorems might you use in justifying your answer to 1 (*b*) ?

3. What assumptions help you in the construction of a perpendicular to a given line from a given point ?

4. How would you test a quadrilateral to see if it were (*a*) a parallelogram, (*b*) a rhombus, (*c*) a rectangle, (*d*) a square?

5. What assumption and what theorem do you use in proving that the exterior angle of a triangle formed by producing one of the sides is equal to the sum of the two interior opposite angles ?

6. What assumption do you bear in mind when testing to see if the two non-coincident arms of two adjacent angles are in the same straight line ?

7. What is meant by a regular figure ?

8. How would you determine the magnitude of (*a*) the interior, (*b*) the exterior angle of a regular figure ?

9. What is the magnitude of the interior angle of (*a*) a regular pentagon, (*b*) a regular hexagon, (*c*) a regular octagon (an eight-sided figure) ?

10. The side **BA** of a cyclic quadrilateral **ABCD** is produced to **E**, $\angle$**DAE**$=40°$. Find the magnitude of the other angles of the figure.

11. With what lines of symmetry are you acquainted ? (Name the symmetrical figures and their lines of symmetry.)

12. Consider:—(*a*) an equilateral triangle, (*b*) a square, (*c*) a regular pentagon, (*d*) a regular octagon, (*e*) a regular figure with twenty sides. What can you say as to the symmetry of each of these figures ?

CHAPTER IV

Arithmetic

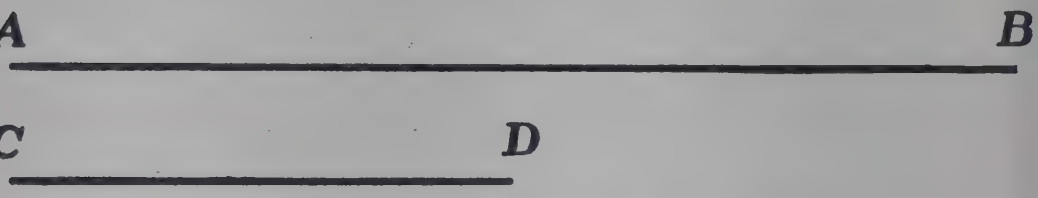
54. There are two ordinary ways of comparing quantities :—

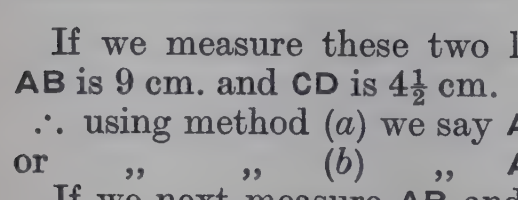
(*a*) We can say **by how much** the larger is greater than the smaller.

(*b*) We can say **how many times** the larger contains the smaller.

Of these two methods, the first depends on the special units we use, whereas the second is quite general and always gives the same result whatever unit is used.

Let us consider two lines, **AB** and **CD**.





If we measure these two lines in centimetres, we find **AB** is 9 cm. and **CD** is $4\frac{1}{2}$ cm.

$\therefore$ using method (*a*) we say **AB** is $4\frac{1}{2}$ cm. longer than **CD**,

or ,, ,, (*b*) ,, **AB** is twice **CD**.

If we next measure **AB** and **CD** in inches, we find **AB** is 3·54 in. and **CD** is 1·77 in.

i.e. using method (*a*) we say **AB** is 1·77 in. longer than **CD**,

but ,, ,, (*b*) we *still* say **AB** is twice **CD**.

55. The method of comparing two quantities by finding **how many times** one quantity contains the other is called finding the **ratio** of one quantity to the other.

The ratio of 12 to 6 is the number of times 12 contains 6.

This is written either $12 : 6$ or $\frac{12}{6}$. If we reduce this to its lowest terms, we get

$$12 : 6 = 2 : 1$$

or

$$\frac{12}{6} = \frac{2}{1}.$$

The ratio reduced to its lowest terms gives the number by which the second term of the ratio must be multiplied to give the first term.

56. There are two points that we must never forget about ratio questions.

(i) We can find how many times one quantity contains another only if they are the **same kind** of quantities; *e.g.* there is, obviously, no sense in saying 10 men are twice £5, although we can say, either 10 men are twice 5 men, or £10 is twice £5.

(ii) **A ratio is a number.**—We cannot say that a *number of times* is so many inches or centimetres or persons or pounds, and we know that a ratio is a number of times; *e.g.* the ratio of 10 cm. to 17 cm. is 10 : 17, and it is exactly the same as the ratio of 10 cows to 17 cows which is also 10 : 17.

57. EXAMPLE 1.—*Can we compare* 6*d.* to 5*s.* *?*—We cannot do so as they are now written, as one quantity is in pence and the other in shillings, but they are the *same kind* of quantities, so we can express them both in pence or both in shillings, and then find the ratio.

In pence this gives the ratio of 6d. to 60d. which is $\frac{6}{60}$ or $\frac{1}{10}$.

In shillings ,, ,, $\frac{1}{2}$s. ,, 5s. ,, $\frac{\frac{1}{2}}{5}$ or $\frac{1}{10}$.

Examples IVa

Lower.

1. Find the ratio of :—

   (*a*) 26 ft. to 2 ft. (*c*) 26 ft. to 1 ft.
   (*b*) 26 ft. to 13 ft. (*d*) 26 ft. to 2 yd.

2. What is the ratio of 5 lb. 2 oz. to 13 lb. 6 oz. ?

3. Could you find the ratio of 10 sq. ft. to 5 ft. ? Give the reason for your answer.

4. Could you find the ratio of 10 cm. to 5 in. ? Give the reason for your answer, and if it is *yes*, say how you would find the ratio.

5. Could you find the ratio of the amount of work done by a man working in a factory to the amount of work done by a woman working in the same factory for the same time ?

Middle.

6. Find the ratio of 42 sq. ft. to 12 sq. yd., also the ratio of 42 ft. to 12 yd. Which gives the larger ratio ? Try to explain why the answers are different.

7. Could you find the ratio between the ratio of £5 to £2 and the ratio of 12 men to 3 men ? Give reasons for your answer.

Higher.

8. If you wanted to describe to a younger brother or sister what the word *ratio* means, how would you do it ?

9. What is the ratio of 15 to 21 ? How could you find the number which has the same ratio to 24 that 15 has to 21 ? (*Note the last part of paragraph* 55.)

10. What is the number by which 17 must be multiplied to give 13 ?

11. What number has the same ratio to 1, that 1 has to 10 ?

58. Often we find quantities so related that alteration in one alters the other.

E.g. if we go shopping we can say, in general, the more we buy the more money we spend ;
or we can say, in general, the older a child is, the taller it is ;
or the larger a room, the greater number of people can sit in it ;
or the larger the audience at a concert, the greater the profits will be ; and so on.

Now let us look more carefully at these examples.

(*a*) *The more we buy the more money we spend.*—Is this true ? In order that this statement may be true, we must refer to buying the same quality of the same goods and at the same price, which probably means at the same shop and at the same time. In other words, the conditions under which the buying is done must remain the same, then we can say definitely that, if a given number of yards are bought for a certain sum of money, twice the number of

yards will cost twice the sum of money, three times the number of yards will cost three times the sum of money, and so on. The mathematical way of expressing this is: **The ratio of the number of yards bought to the given number of yards is the same as the ratio of the cost to the given sum of money,** or **the amount of money spent alters in the same ratio as the number of yards bought.**

(*b*) *The older a child is the taller it is.*—It is true that an increase in age for children is generally accompanied by an increase in height, but there are no conditions under which we can say that a child of eight is necessarily twice as tall as a child of four, or a child of twelve three times as tall as a child of four, *i.e.* a child's height does not increase in the same ratio as its age.

Be careful to notice whether the questions in the following examples are of type (*a*) or (*b*).

Examples IVb (*oral*)

Under the same conditions.

1. If a train runs 120 miles in 4 hours, how far would it run in (i) 8 hours, (ii) 12 hours, (iii) 2 hours, (iv) 6 hours ?
2. If a mustard plant grows 2 in. in three weeks, would it have grown 34⅔ in. in a year ?
3. If 7 lb. of sugar can be bought for 3s. 6d., what amounts can be bought for (i) 7s., (ii) 14s., (iii) 1s. 9d., (iv) 8s. 9d., (v) £1, 2s. 9d. ?

59. Examples IVb can be worked mentally, but in many cases where we make use of equal ratios the numbers do not work out so simply; the method, however, is the same.

EXAMPLE 1.—*It costs* £1, 15*s. to carpet a flight of* 14 *stairs; what will it cost to cover* 5 *such stairs with the same kind of carpet?*

We want to find a cost: we write down first the complete statement we are told about one cost, and beneath it the question we are asked.

Statement: Carpet for 14 stairs costs £1, 15s.
Question: „ „ 5 „ „ ?

The ratio of the new number of stairs to the old number $=\frac{5}{14}$.

$\therefore$ the new cost is $\frac{5}{14}$ of the old cost;

i.e. new cost $= \frac{5}{14}$ of 35s.

$= \frac{25}{2}$s.

$=$12s. 6d;

i.e. cost of carpeting 5 stairs is 12s. 6d.

EXAMPLE 2.—*If* $3\frac{3}{4}$ *yd. of silk cost* 25s., *how many yards can be bought for* 17s. 6d. *?*

We want to find a number of yards, therefore we write down the complete statement we are told about one number of yards, and beneath that the answer to the question.

Statement: 25s. will buy $3\frac{3}{4}$ yd. of silk.

Answer to Question: 17s. 6d. „ „ $3\frac{3}{4} \times \frac{17\frac{1}{2}}{25}$ yd. of silk;

i.e. $\frac{\overset{3}{\cancel{15}}}{4} \times \frac{\overset{7}{\cancel{35}}}{2 \times \cancel{25}_{\cancel{5}}}$ yd. of silk

$= \frac{21}{8}$ yd. of silk

$= 2\frac{5}{8}$ yd. of silk;

i.e. 17s. 6d. will buy $2\frac{5}{8}$ yd. of silk.

[*N.B.*—We wrote down the number of yards we knew, and multiplied it by the ratio of the new amount spent to the old amount spent. We have thus found the number of yards that has the same ratio to $3\frac{3}{4}$ yd. that 17s. 6d. has to 25s.]

Examples IVc

1. If eggs cost 2s. 9d. a dozen, what is the price of 30 eggs ?
2. Eighteen tons of coal costs £52, 4s., what is the cost of 45 tons ?
3. Sugar costs 6s. 5d. a stone, what is the cost of 12 cwt. ?
4. Twenty yards of a certain cretonne cost £3, 15s., what will it cost to curtain 17 windows with this cretonne, allowing 8 yd. per window ?

5. The amount paid out per week in wages to 15 men is £40, 10s., how much would be paid to 23 men earning wages at the same rate ?

6. What is the price per dozen of oranges when 1000 oranges cost £5, 4s. 2d. ?

7. What is the cost of garaging a car for 15 weeks if it costs £2, 3s. for 4 weeks ?

8. What is the value of 15 cases of tea each containing 7 lb. if 1 cwt. of the tea is worth £21 ?

9. One pound of nails costs 2s. 1d; if there are 300 nails in 1 lb., what is the price of 3 dozen nails ?

10. A club has 300 members in a certain year, and their subscriptions yield an income of £472, 10s. to the club. It is decided the following year to increase the membership to 350, and an entrance fee of £2 is charged to all new members in addition to the annual subscription. What is the new total income of the club ?

60. We shall now consider one of the special cases where equal ratios are used.

People who have money do not, as a rule, keep it in a safe. They **invest** it ; that is, the money is lent or entrusted to a business firm or a bank which uses it for trading ; the bank pays the owner of the money a certain sum each year for the use of the money, this sum is called the **Interest** on the money. If one man lends another money, the borrower pays back the money lent and, in addition, an amount of interest such as a bank would have paid had the money been invested. From the borrower's point of view the interest is the extra money he has to pay for the convenience of having had the borrowed money just when he needed it.

EXAMPLE 1.—*If the interest on* £100 *for one year is* £4, *what would be the interest on* £905, 5*s. for the same time ?*

On £100 for 1 year interest is £4.

„ £905, 5s. „ „ $£4 \times \frac{905\frac{1}{4}}{100}$

$$= £\frac{\overset{1}{\cancel{4}} \times 3621}{\underset{100}{\cancel{400}}}$$

$$= £36{\cdot}21$$

$$= £36,\ 4s.\ 2d.$$

[*N.B.*—The bank does not deal in fractions of a penny. When paying out money it pays to the nearest penny below any fraction, when charging a customer it demands the nearest penny above any fraction.]

EXAMPLE 2.—*If the interest on* £100 *for* 1 *year is* £5, *on what sum of money would the interest for the same time be* £32, 10*s.*?

Here the answer is to be a sum of money invested, so the statement is arranged so that the money invested is the last term.

£5 is the interest in 1 year on £100.

£32, 10s. „ „ „ $£100 \times \frac{32\frac{1}{2}}{5}$

$= £\frac{100 \times 65}{10}$

$= £650.$

61. The sum of money invested in a bank or other business is called the **Principal.**

The interest on £100 for one year is called the **rate per cent. per annum.** Instead of the Latin abbreviation per cent., it is usual to write the symbol %.

EXAMPLE 1.—*At what rate* % *per annum would the interest on* £5640 *be* £197, 8*s. in one year?*

We are asked to find rate %, that is, the interest on £100.

On a principal of £5640 the interest is £197·4.

On a principal of £100 the interest is $£197{\cdot}4 \times \frac{100}{5640}$

$= £\frac{1974}{564}$

$= £\frac{987}{282}$

$= £\frac{7}{2}$

$= £3\frac{1}{2}$

$\therefore$ rate of interest is $3\frac{1}{2}$ %.

EXAMPLE 2.—*What is the interest on* £1850 *for* 4 *years at* $5\frac{1}{2}$% *per annum?*

We are asked to find the interest for more than one year ; the interest here depends on (*a*) the rate, (*b*) the ratio of the principal to £100, (*c*) the ratio of the time to one year.

On £100 for 1 year interest = £5½.

$$\text{,, £1850 ,, 4 years ,, } = £\frac{11}{\cancel{2}} \times \frac{\cancel{4}}{1} \times \frac{37\ \cancel{1850}}{\cancel{100}\ \cancel{2}}$$

$$= £407.$$

Interest is £407.

62. Interest calculated in this manner (61, *Example* 2) is called **Simple Interest.**

Examples IVd

1. Find the Simple Interest on £895 for 3 years at 5% per annum.
2. At what rate % is £261, 13s. 4d. the Simple Interest on £7850 for 1 year ?
3. Find the Simple Interest on £1255, 10s. 6d. for 11 years at 2½% per annum.
4. At what rate % is the Simple Interest on £675 for 5 years £236, 5s. ?
5. What is the Simple Interest on £8000 for 3 months at 2½% per annum ?
6. Find the Simple Interest on £540 for 18 months at 3⅓% per annum.
7. Which is greater, and by how much, the Simple Interest on £5750 for 2 years at 5% per annum or the Simple Interest on £6500 for 1 year 6 months at 4¾% per annum ?
8. If £715, 10s. is the Simple Interest on £7950 for 4 years, find the rate % per annum.
9. What is the Simple Interest on £6784, 10s. for 73 days at 5½% per annum ?
10. At what rate % per annum is the Simple Interest on £2760 for 3 years 4 months £345 ?

63. If the principal and interest are added, the result is called the **Amount,** *e.g.* :—

(i) The amount of £100 in 1 year at 5% per annum is £105.

(ii) The amount of £100 for 3 years at 4% per annum is £112, *i.e.* £100 principal + £12 interest.

EXAMPLE 1.—*What would be the amount of* £765 *in* 7 *years at* $3\frac{1}{3}$ *per cent. per annum Simple Interest?*

In this case we work out the Simple Interest for the given time and add it to the £765 principal.

On £100 for 1 year interest is £$3\frac{1}{3}$.

„ £765 „ 7 years „ £$\frac{10}{3} \times \frac{7}{1} \times \frac{765}{100}$

$$= £\frac{7 \times \overset{51}{\cancel{255}}}{\underset{2}{\cancel{10}}}$$

$$= £\frac{357}{2}$$

$$= £178, 10s.$$

$\therefore$ The Amount = £765 + £178, 10s.

= £943, 10s.

Note.—The interest increases in the same ratio as the time. The amount **does not** increase in the same ratio as the time.

Consider £100 invested at $4\frac{1}{2}\%$ per annum.

In 1 year Simple Interest is £$4\frac{1}{2}$ and amount is £$104\frac{1}{2}$.

„ 2 years „ „ £9 „ „ „ £109.

„ 3 „ „ „ £$13\frac{1}{2}$ „ „ „ £$113\frac{1}{2}$, etc.

As the time increases, the amount increases, but they do not increase in the same ratio.

64. When rate, principal, and amount are given, the time can be found by either of the following methods:—

In how many years would £160 *amount to* £200 *at* 5% *per annum Simple Interest?*

Method (*a*)—

A principal of £160 produces an amount of £200.

A principal of £100 produces an amount of £$\overset{25}{\cancel{200}} \times \frac{\overset{5}{\cancel{100}}}{\underset{2}{\cancel{160}}}$

$$= £125.$$

For the required time amount of £100 is £125.
" " " interest on £100 is £25.
Interest on £100 is £5 for 1 year (since rate is 5% per annum).
∴ interest on £100 is £25 for 5 years.
£160 amounts to £200 in the same time, *i.e.* 5 years.

Method (*b*)—

The given amount is £200.
" " principal " £160.
∴ the interest " £40.

On £160 for the required time the interest is £40.

On £100 for the required time the interest is $£\not{4}\not{0} \times \frac{25}{\not{1}\not{6}\not{0}_{\not{4}}}$

$= £25.$

Interest on £100 is £5 for 1 year.
" " £25 " 5 years.
∴ required time is 5 years.

65. If rate, time, and amount are given, to find the principal it is necessary to find the amount of £100 for the given time and rate, as follows :—

What sum of money would amount to £640 *in* 7 *years at* 4% *per annum Simple Interest ?*

The interest on £100 for 7 years at 4% per annum =£28 ; *i.e.* under the given conditions the amount of £100 is £128.

An amount of £128 is obtained from a principal of £100.
An amount of £640 is obtained from

a principal of . . $£100 \times \frac{\overset{5}{\not{6}\not{4}\not{0}}}{\not{1}\not{2}\not{8}}$

$= £500.$

The required sum of money is £500.

66. In all the examples we have considered so far, we have dealt with money in pounds sterling ; the same reasoning applies if any other standard coin be taken as unit.

EXAMPLE 1.—*Find the Simple Interest on* 5875 *francs for* 6 *years at* $4\frac{1}{2}\%$ *per annum.*

On 100 francs for 1 year interest is $4\frac{1}{2}$ francs.

On 5875 francs for 6 years interest is $\frac{\overset{3}{\cancel{9}}}{\cancel{2}} \times \frac{\overset{3}{\cancel{6}}}{1} \times \frac{5875}{\underset{20}{\cancel{100}}}$ francs

$= 9 \times 176{\cdot}25$ francs

$= 1586{\cdot}25$ francs.

Required interest is 1586·25 francs.

EXAMPLE 2.—At what rate % would \$1895 amount to \$2274 in 5 years ? (\$ is the sign for dollars.)

A principal of \$1895 produces an amount \$2274.

A principal of \$100 produces an amount $\overset{6}{\cancel{\$2274}} \times \frac{\overset{20}{\cancel{100}}}{\underset{379}{\cancel{1895}}}$

$= \$120.$

At the required rate, amount of \$100 is \$120 ;
i.e. interest on \$100 for 5 years is \$20 ;
∴ „ „ „ 1 year is \$4.

Required rate is 4%.

EXAMPLE 3.—What sum of money will amount to 2350 francs in 5 years at $3\frac{1}{2}\%$ per annum ?

Interest on 100 francs for 5 years at $3\frac{1}{2}\% = 17{\cdot}5$ francs ; *i.e.* under the given conditions the amount of 100 francs is 117·5 francs.

An amount of 117·5 francs is obtained from a principal of 100 francs.

An amount of 2350 francs is obtained from a principal of $\overset{20}{\cancel{100}} \times \frac{\overset{100}{\cancel{2350}}}{\underset{3{\cdot}5}{\cancel{117{\cdot}5}}}$ francs

$= 2000$ francs.

Required sum of money is 2000 francs.

Examples IVe

Lower.

1. At what rate per cent. is the interest on £550 for 1 year £22 ?
2. If the interest on £100 is £18, what will be the interest on £420, 15s. for the same time ?
3. If the interest on £100 is £5 per annum, what sum of money has £240 interest per year ?
4. What is the interest on £719, 6s. for 1 year at $4\frac{1}{2}\%$ per annum ?
5. What is the rate per cent. if the interest on £960 for 5 years is £160 ?
6. In how many years will the interest on £100 be itself £100 at 5% per annum ?
7. What is the Simple Interest on £895, 13s. 6d. for 5 years at $3\frac{1}{3}\%$ per annum ?
8. Find the amount of £520, 14s. for 7 years at 4% per annum.
9. Find the rate % per annum when the interest on £2050 is £322, 17s 6d. in 3 years.
10. What is the Simple Interest on 7875 francs for 3 years at 4% per annum ?
11. Find the interest on \$6950 for 4 years at $3\frac{1}{2}\%$ per annum.

Middle.

12. Find the interest on £1025, 15s. for 7 years at $4\frac{1}{2}\%$ per annum.
13. What is the amount of £190 in 6 years at $2\frac{1}{4}\%$ per annum ?
14. At what rate % will £8550 amount to £12,255 in 10 years ?
15. What is the interest on £5275 for 3 years at 4% per annum ?
16. What sum of money will amount to £9371, 5s. in 5 years at $4\frac{1}{2}\%$ per annum ?
17. What is the principal which amounts to £3422, 10s. in 7 years at $3\frac{1}{3}\%$ per annum ?
18. At what rate per cent. per annum will £800 amount to £900 in 4 years ?

Higher.

19. What sum of money will have the same interest in 4 years, that £2400 has in $3\frac{1}{2}$ years, at 5% per annum in each case ?

20. What sum of money will amount to £194, 8s. at 4% per annum during the same time that £500 amounts to £550 at 5% per annum ?

21. £2590 is invested for 12 years ; for the first 5 years the rate of interest is 3% per annum, for the remaining 7 years it is 5% per annum. Would it have been better to have invested the money for the whole time in a business paying 4% per annum ?

22. Which gives the greater interest, and by how much, £870 invested for 10 years at 4% per annum, or £2760 invested for 3 years at 5% per annum ?

23. What is the rate per cent. per annum if 9975 francs amounts to 10,507 francs in 1 year ?

24. What principal will amount to $42,330 in 7 years at $3\frac{1}{2}$% per annum ?

25. What is meant by *Invest, Principal, Rate, Interest, Amount?*

67. When goods are bought at one price and sold at another, the **difference** between the two prices gives the **profit** (gain) or **loss**. There is a gain when the selling price (S.P.) is greater than the buying price (B.P.). There is a loss when the S.P. is smaller than the B.P.

Profit and **loss** are calculated with reference to the **outlay,** that is, the buying price or cost price (C.P.) of the goods.

EXAMPLE 1.—*If goods bought for* £540 *are sold for* £630, *on what outlay would there be a gain of* £1000 ?

$$\text{Gain} = £630 - £540$$
$$= £90$$

Method (*a*)—

£90 is the gain on an outlay of £540.

£1000 ,, ,, ,, $£\cancel{540} \times \frac{100\cancel{0}}{\cancel{90}}^{60}$

$= £6000.$

Required outlay is £6000.

Method (*b*)—

$$\text{Gain} = \frac{90}{540} \text{ of outlay}$$
$$= \tfrac{1}{6} \quad ,,$$

Required gain = £1000,

$\therefore$ required outlay = £6000.

68. It is very usual to express profit or loss as a fraction of the outlay, as in paragraph 67, Example 1, Method (*b*), when the gain is $\frac{1}{6}$ of the outlay. Had the outlay been 100, the gain would have been $\frac{1}{6}$ of 100 or $16\frac{2}{3}$ per cent. The gain or loss is nearly always expressed as percentage gain or loss, and then, if the rates of gain or loss on a number of transactions have to be compared, a comparison is more easily made.

EXAMPLE 1.—*Is it better to buy tea at* £13, 1*s.* 4*d. per cwt. and sell at* £14 *per cwt., or to buy at* 2*s.* 6*d. per lb. and sell at* 2*s.* 8*d. per lb. ?*

1st Case—

B.P.=£13, 1s. 4d. ($£13\frac{1}{15}$).
S.P.=£14.
Gain=18s. 8d. ($£\frac{14}{15}$).

Gain is $\frac{\frac{14}{15}}{13\frac{1}{15}}$ of outlay.

$$\text{Gain is } \frac{\cancel{14}^{1}}{\cancel{196}} \text{ of } \cancel{100}^{50}\%$$

$$= \frac{50}{7}\%$$

$$= 7\tfrac{1}{7}\%.$$

2nd Case—

B.P.=2s. 6d.
S.P.=2s. 8d.
Gain=2d.

$$\text{Gain is } \frac{2}{30} \text{ of outlay}$$

$$= \frac{1}{\cancel{15}} \text{ of } \cancel{100}^{20}\%$$

$$= 6\tfrac{2}{3}\%.$$

It is better to buy at £13, 1s. 4d. and sell at £14 per cwt.

EXAMPLE 2.—*A house is bought for* £2500 *and sold for* £1800. *What is the loss* % *?*

$$\begin{aligned} \text{B.P.} &= £2500 \\ \text{S.P.} &= £1800 \\ \text{Loss} &= £700 \\ \text{Loss} &= \frac{700}{2500} \text{ of outlay} \\ &= \frac{7}{25} \text{ of } 100\% \\ &= 28\%. \end{aligned}$$

The house is sold at a loss of 28%.

EXAMPLE 3.—*If a man loses* £75 *by selling a house for* £1800, *what was his loss* % *?*

We must remember the important fact noted in paragraph 67. **Gain** and **loss** are each **calculated on** the **buying price.**

Here we are not actually told the B.P., so we must calculate what it would be.

$$\begin{aligned} &\text{S.P. is } £1800 \\ &\text{Loss is } £75 \\ \therefore\ &\text{B.P. is } £1875 \\ &\text{Loss is } \frac{75}{1875} \text{ of outlay} \\ &= \frac{7500}{1875}\% \\ &= 4\%. \end{aligned}$$

EXAMPLE 4.—*A man gained* 12% *by selling his car for* £259. *What did he pay for it ?*

A gain of 12% means that when 100 is standard B.P. then 112 „ S.P.

i.e. B.P. is $\frac{100}{112}$ of S.P.

$$\begin{aligned} &= \frac{\cancel{100}^{25}}{\cancel{112}} \text{ of } £\cancel{259}^{37} \\ &= £\frac{925}{4} \\ &= £231, 5s. \end{aligned}$$

He paid £231, 5s. for the car.

EXAMPLE 5.— *An oak table is sold for* £17, 8*s. at a loss of* 13%. *What was its cost price?*

A loss of 13% means that when 100 is standard C.P.
then 87 ,, S.P.

i.e. C.P. is $\frac{100}{87}$ of S.P.

$$= \frac{100}{87} \text{ of £17, 8s.}$$

$$= £\frac{100}{87} \times \frac{87}{5}$$

$$= £20$$

Table costs £20.

Examples IVf

Lower.

1. What is the percentage gain if an article bought for 15s. is sold for 16s. 6d.?

2. Which is the more profitable transaction; to buy smal houses at £450 and sell them at £480 each, or to buy large houses at £1600 and sell them at £1700 each? (Find per cent. gain in each case and compare results.)

3. The score of a cricket innings is as follows: 0, 141, 151, 4, 100, 4, 26, 42, 14, 1, 1, extras 10. Find what percentage of the total score was due to the highest score of the innings.

4. If the price of railway tickets has been increased 50%, find to the nearest penny the old price of a ticket which now costs 15s. 7d.

5. If a reduction of 10% is made on the marked price of furs, what is the marked price of furs for which sixty guineas are paid?

6. What is the gain % if potatoes bought for 17s. 6d. per cwt. are sold at 2d. per lb.?

7. What is the gain or loss % if eggs are bought at the rate of 2s. for 20 and sold at $1\frac{1}{4}$d. each?

8. Sixty lb. of strawberries are bought for 25s. It is then found that 5% of them have been spoiled in carriage. The remainder is sold at the rate of 7d. per lb. What is the total gain or loss on the transaction? Express this as a percentage.

9. Tea is bought at £14 per cwt. 50% is sold at 3s. 6d. per lb., 10% at 3s. 9d. per lb., the rest is sold for £4. What is the gain or loss per cent.?

10. Tomatoes are bought wholesale for 10d. per lb. and sold at a gain of 5%. When the wholesale price falls 2d. per lb., the retail price is also reduced by 2d. per lb. What is the new gain per cent. ?

Middle.

11. One cwt. of coffee is bought at 2s. 8d. per lb. ; what is the gain or loss per cent. on the whole if 12 lb. are spoiled, and the rest is sold at 3s. 3d. per lb. ?

12. Eggs are bought at 4s. a score and sold at $2\frac{3}{4}$d. each ; what is the gain or loss per cent. ?

13. The population of a district increases by 640 during a year ; at the end of the year the population is 18,000. What is the percentage increase ?

14. Salt is composed of 23 parts by weight of sodium and 35·5 parts of chlorine. What is its percentage composition ?

15. The selling price of potatoes falls from 5d. to 4d. per lb., yet the tradesman who formerly made 20% profit on his outlay now makes 25%. Find to the nearest halfpenny the wholesale price per cwt. at each rate. Compare the percentage fall in price at the wholesale and retail rates.

16. One metre = 39·37 in. Express 1 kilometre as a percentage of 1 mile.

17. Tea is bought in London at 2s. per lb. It is sent to Paris at a cost of 5s. 6d. per cwt. At what price in francs and centimes per kilogram must it be sold to give a profit of 20% ? (Consider 112 francs equivalent to 20s. and 2·2 lb. equivalent to 1 kilogram.)

18. Eighteen lb. of tea at 2s. 8d. per lb. are mixed with 24 lb. at 3s. At what price per lb. must the mixture be sold to gain 20% ?

Higher.

19. At what price must a dealer mark goods worth £3, 3s. if he is to make a profit of 12% by selling them at 20% below their marked price ?

20. Express 1 litre as a percentage of 1 gallon given :—

Weight of 1 litre of water = 2·204 lb.
Weight of 1 cu. ft. of water = 1000 oz.
10 cu. ft. = 62·5 gallons.

21. A sells goods to B at a profit of 20%. B sells to C at a profit of 12%, and C sells to D for £290, 6s. 1d. at a profit of 8%. What did the goods cost A ?

22. A tradesman sells certain articles at $7\frac{1}{2}$d. each and makes a profit of 25%. The cost price is increased 10% and he now sells for $8\frac{1}{4}$d. each. What is the difference in his gain per cent. ?

69. EXAMPLE 1.—*If* 10 *lb. of tea cost* 27*s.* 6*d., what will be the cost of* 40 *lb. of the same tea?*

Here we see that when the amount of tea is *increased*, the total cost will be *increased*. There is four times as much tea, so the cost will be four times as much.

So, if we multiply the amount of tea by 8, we must multiply the cost by 8; if we multiply the amount of tea by x, we must multiply the cost by x.

We express this by saying that the cost **varies directly** as the amount of tea.

As illustrations of this :—

(1) When the rate and time remain the same, the Simple Interest on a sum of money varies directly as the principal invested.

(2) When the rate and principal remain the same, the Simple Interest varies directly as the time.

Give two other simple examples in which one quantity varies directly as another.

70. If you look back at the work you have already done, you will see that in each ratio sum one quantity varied directly as one other quantity or as several other quantities. This is not the only kind of ratio sum. Consider the following example.

EXAMPLE 1.—*If* 4 *men can dig a garden in* 5 *days, how long will it take* 2 *men to dig it, working at the same rate ?*

Here we see that when the number of men is *decreased* the length of time will be *increased*. There are only half the number of men working, so they will take twice the time to dig the garden.

So, if we multiply the number of men by 4, we must divide the time taken by 4; if we multiply the number of men by x, we must divide the time taken by x.

We express this by saying that the time taken **varies inversely** as the number of men.

Give two simple examples in which one quantity varies inversely as another.

EXAMPLE 2.—*If* 15 *men can mend a piece of road* 9 *ft. wide and* 25 *yd. long in* 18 *days, how long would it take* 20 *men working at the same rate to mend a similar stretch of road* 10 *ft. wide and* 20 *yd. long ?*

The number of days varies directly (*a*) as the width of road.
" " " (*b*) " length "
" " inversely " number of men.

15 men mend road 9 ft. wide by 25 yd. long in 18 days.

20 men mend road 10 ft. wide by 20 yd. long in

$$18 \times \frac{20}{25} \times \frac{10}{9} \times \frac{15}{20} \text{ days.}$$

$$= 12 \text{ days.}$$

Required time is 12 days.

EXAMPLE 3.—*A garrison of* 150 *men have supplies which will last* 8 *weeks. Another* 50 *men join the garrison and bring with them more supplies which represent an increase of* 50% *on what is already in the garrison. How many weeks will the supplies now last ?*

The time the supplies will last varies directly as amount of supplies, but inversely as the number of men.

Supplies represented by 100 last 150 men for 8 weeks.
Supplies represented by 150 last 200 men for

$$\overset{4}{\cancel{8}} \times \frac{\overset{3}{\cancel{150}}}{\underset{4}{\cancel{200}}} \times \frac{\overset{3}{\cancel{150}}}{\underset{2}{\cancel{100}}} \text{ weeks}$$

$$= 9 \text{ weeks.}$$

Required time is 9 weeks.

Examples IVg

Lower.

1. If 20 men take 15 days to dig a trench 18 ft. broad 5 ft. deep and 2 miles long, what length of trench 4 ft. deep and 12 ft. broad will 140 men dig in 2 days ?

2. In how many years will the Simple Interest on £255 be £34 at 5% per annum ?

3. On what principal is the Simple Interest for $7\frac{1}{2}$ years at $4\frac{1}{2}$% per annum £891 ?

4. A garrison of 150 men has rations for 25 days. After 4 days the garrison is increased by 30 men, so the rations are cut down $\frac{1}{6}$. Four days later 10 men leave the garrison. For how many days longer have the remainder rations at the reduced rate ?

Middle.

5. A farmer considers certain lands will provide pasturage for 54 sheep for 32 days. After 12 days he removes the sheep and replaces them by 10 cows. How long can the cows be left on this land if 12 cows need the same pasturage as 27 sheep ?

Higher.

6. Refreshments are provided at a bazaar for 450 people. The charge is to be 9d. per head. The day is very wet and 10% of the food provided is spoilt and has to be replaced at a cost of £1, 2s. 6d. Only 360 people are served. What price should they be charged in order that the anticipated profit may be cleared ? (Answer to the nearest penny.)

7. An employer pays £175 in wages to 25 men who work for 14 days at 8 hours a day. He has a similar piece of work twice as great which must be done in 12 days ; he employs 50 men and agrees to pay them at a 20% increased rate for any overtime beyond the eight hours day. What does he pay out in wages for the second piece of work ?

Miscellaneous Examples involving Ratio

71. Example 1.—*If* A *can give* B 10 *yd. and* C 15 *yd. in a quarter mile race, by how much can* B *beat* C *in* 300 *yd.?*

While A runs 440 yd. B runs 430 yd. and C runs 425 yd.

While B runs 430 yd. C runs 425 yd.

„ B „ 300 „ C „ $425 \times \frac{300}{430}$ yd.

$$= \frac{12750}{43} \text{ yd.}$$

$$= 296\tfrac{22}{43} \text{ yd.,}$$

$\therefore$ B can give C $(300 - 296\frac{22}{43})$ yd. in 300 yd. ;

i.e. B „ C $3\frac{1}{2}$ yd. (to nearest $\frac{1}{2}$ yd.) in 300 yd.

(Note the swifter runner completes the whole course.)

EXAMPLE 2.—*If* A *can beat* B *by* 10 *yd. in* 100 *yd. race and* B *can beat* C *by* 15 *yd. in* 100 *yd. race, by how much can* A *beat* C *in* 100 *yd. race?*

We can compare the distances run by A and B,
and we ,, ,, ,, C ,, B.

While A runs 100 yd., B runs 90 yd.

We must find how far C runs while B runs 90 yd.

While B runs 100 yd., C runs 85 yd.

,, B ,, 90 ,, C ,, $\frac{85 \times 90}{100}$ yd.

$$= \frac{153}{2} \text{ yd.}$$

$$= 76\tfrac{1}{2} \text{ yd.};$$

i.e. C runs $76\frac{1}{2}$ yd. while B runs 90 yd., *i.e.* while A runs 100 yd.,

$\therefore$ A can give C $(100 - 76\frac{1}{2})$ yd. in 100 yd. race;

i.e. A ,, C $23\frac{1}{2}$,, 100 ,,

72. EXAMPLE 1.—*Two clocks are set at* 10 *a.m. on Saturday. One keeps correct time, the other loses* 15 *minutes in* 3 *days. What time does the correct clock indicate when it is* 3 *p.m. on Tuesday by the slow clock?*

10 a.m. Saturday to 3 p.m. Tuesday is 77 hr.

The slow clock loses 15 min. in 3 days,

,, ,, ,, $\frac{1}{4}$ hr. ,, 72 hr.;

i.e. the slow clock goes $71\frac{3}{4}$ hr. in 72 hr.

While the slow clock goes $71\frac{3}{4}$ hr., the correct clock goes 72 hr.

While the slow clock goes 77 hr., the correct clock goes

$$72 \times \frac{77}{71\frac{3}{4}} \text{ hr.}$$

$$= \frac{72 \times 77 \times 4}{287} \text{ hr.}$$

$$= \frac{2772 \times 8}{287} \text{ hr.}$$

$$= \frac{22176}{287} \text{ hr.}$$

= 77 hr. 16 min. 6 sec. (to nearest second).

The correct time is 16 min. 6 sec. past 3 p.m. on Tuesday.

EXAMPLE 2.—*Two clocks are set correctly at noon on Sunday. One gains* 10 *min. in* 96 *hr. and the other loses* 15 *min. in* 72 *hr. What will the second clock indicate when the first is at* 10 *a.m. on Wednesday?*

In questions of this kind it is wise to compare in turn the time indicated on each clock with the correct time.

From noon Sunday to 10 a.m. on Wednesday is 70 hr.

The fast clock gains $\frac{1}{6}$ hr. in 96 hr.
The slow clock loses $\frac{1}{4}$,, 72 ,,

When the fast clock shows $96\frac{1}{6}$ hr. to have passed, by the correct time 96 hr. have passed.

(*a*) When the fast clock shows 70 hr. to have passed, by the correct time $96 \times \frac{70}{96\frac{1}{6}}$ hr. have passed;

i.e. $\frac{96 \times 420}{577}$ hr.

When by the correct time 72 hr. have passed, the slow clock shows $71\frac{3}{4}$ hr. have passed.

(*b*) When by the correct time $\frac{96 \times 420}{577}$ hr. have passed, the slow clock shows $71\frac{3}{4} \times \frac{96 \times 420}{577 \times 72}$ hr. have passed.

$$\text{i.e. } \frac{287 \times \cancel{96} \times \overset{140}{\cancel{420}}}{\cancel{4} \times 577 \times \overset{3}{\cancel{72}}} \text{ hr.}$$

$$= \frac{40180}{577} \text{ hr.}$$

$$= 69 \text{ hr. } 38 \text{ min. (to nearest minute)};$$

i.e. the slow clock shows 69 hr. 38 min. to have passed since noon on Sunday.

When the fast clock indicates 10 a.m. on Wednesday, the slow clock indicates 9.38 a.m. (nearest minute).

73. *A bath is filled in* 20 *min. by the cold water when turned on fully, and in* 25 *min. by the hot water when turned on fully. When the waste pipe is open, the bath empties itself in* 10 *min.*

(*a*) *How long will it take to fill the bath with both hot and cold water turned on fully?*

(*b*) *How long will it take to empty the bath if, when it is full, the hot water is turned on fully and the waste pipe is opened?*

(*c*) *How long will it take to empty the bath if, when it is full, both taps are turned on fully and the waste pipe is opened?*

Consider what takes place in 1 min.

The cold water fills $\frac{1}{20}$ of the bath in 1 min.

The hot ,, $\frac{1}{25}$,, ,, 1 ,,

The waste pipe empties $\frac{1}{10}$ of the bath in 1 min.

(*a*) They together fill $(\frac{1}{20}+\frac{1}{25})$ of the bath in 1 min.

$\therefore$ they together fill $\frac{5+4}{100}$,, ,, ,, 1 min.

$\therefore$,, ,, ,, $\frac{9}{100}$,, ,, ,, 1 min.

$\therefore$,, ,, ,, the whole bath in $\frac{100}{9}$ min.

$=11\frac{1}{9}$,,

(*b*) The waste pipe empties $\frac{1}{10}$ of the bath in 1 min.

The hot ,, fills $\frac{1}{25}$,, ,, 1 ,,

$\therefore$ they together empty $(\frac{1}{10}-\frac{1}{25})$ of the bath in 1 min.

$\therefore$,, ,, ,, $\frac{5-2}{50}$,, ,, ,, 1 min.

$\therefore$,, ,, ,, $\frac{3}{50}$,, ,, ,, 1 min.

$\therefore$,, ,, ,, the whole bath in $\frac{50}{3}$ min.

$=16\frac{2}{3}$,,

(*c*) The hot and cold pipes together fill $\frac{9}{100}$ of the bath in 1 min.

The waste pipe empties $\frac{1}{10}$ of the bath in 1 min.

$\therefore$ they together empty $(\frac{1}{10}-\frac{9}{100})$ of the bath in 1 min.

$\therefore$,, ,, ,, $\frac{1}{100}$,, ,, ,, 1 min.

$\therefore$,, ,, ,, the whole bath in 100 min.

74. EXAMPLE 1.—*Two men,* A *and* B, *start a motor garage business.* A *invests* £4000 *and* B *invests* £2500.

(*a*) *In what proportion should they expect to divide the profits?*

$$\text{The ratio of A's investment to B's investment} = \frac{4000}{2500}$$
$$= \frac{8}{5}.$$

∴ ratio of A's profits to B's profits should be 8 : 5;
i.e. if A's profits are represented by 8 units,
B's ,, ,, ,, 5 ,,
i.e. A receives $\frac{8}{13}$, and B receives $\frac{5}{13}$ of the total profits.

(*b*) *If the profits were* £1300 *in the first year, what should each receive?*

A should receive $\frac{8}{13}$ of £1300, *i.e.* £800.
B ,, ,, $\frac{5}{13}$ of £1300, *i.e.* £500.

(*c*) *During the second year all the management of the business is left to* B, *and for this he is paid a salary of* £400 *out of the profits. In the second year the profits are* £3000: *how will this be divided?*

B receives £400 salary.
There remains £2600 profits to divide.
A receives $\frac{8}{13}$ of £2600, *i.e.* £1600.
B receives his salary + $\frac{5}{13}$ of £2600; *i.e.* £400+£1000.
In second year

A receives £1600 of profits.
B ,, £1400 ,,

EXAMPLE 2.—A *invests* £2000 *in a business for* 5 *months,* B *invests* £1000 *for a year,* C *invests* £1500 *for* 6 *months.* B *receives a salary of* £300 *out of the profits for managing the business. How should profits of* £6500 *be divided?*

In order to compare the amounts invested, each must be expressed as the equivalent sum of money for a standard length of time, say one month.

A invests £2000 for 5 months, which is equivalent to £10,000 for 1 month.

B invests £1000 for 1 year, which is equivalent to £12,000 for 1 month.

C invests £1500 for 6 months, which is equivalent to £9000 for 1 month.

Total is equivalent to £31,000 for 1 month.

Total profits £6500.
B's salary £300.
Profits divided = £6200.

A receives $\frac{10}{31}$ of £6200 = £2000.
B ,, salary + $\frac{12}{31}$ of £6200 = £300 + £2400.
= £2700.
C ,, $\frac{9}{31}$ of £6200 = £1800.

A receives £2000.
B ,, £2700.
C ,, £1800.

75. EXAMPLE 1.—*Tea at 2s. 8d. per lb. is mixed with tea at 3s. 6d. per lb. and the mixture is worth 3s. per lb. In what proportion are the two teas mixed?*

On every lb. of 3s. 6d. tea put in a 3s. mixture there is 6d. loss.
,, ,, 2s. 8d. ,, 3s. ,, ,, 4d. gain.

As the mixture is actually worth 3s., the loss on the dearer tea exactly balances the gain on the cheaper tea.

4 lb. of 3s. 6d. tea causes (4×6)d. loss.
6 ,, 2s. 8d. ,, gives (6×4)d. gain.

These two balance one another,

$\therefore$ ratio in which teas are mixed is

4 lb. at 3s. 6d. : 6 lb. at 2s. 8d.;
i.e. 2 lb. at 3s. 6d. : 3 lb. at 2s. 8d.

EXAMPLE 2.—*Coffee at 3s. per lb. is mixed with chicory at 9d. per lb., and the mixture is sold for 3s. per lb. at a gain of 20%. In what proportion are the coffee and chicory mixed?*

The first step is to find the actual cost price of the mixture.

A gain of 20% means that when the standard C.P. is 100, the standard S.P. is 120;

$$\textit{i.e.}\ \text{C.P. is } \frac{100}{120} \text{ of S.P.}$$
$$= \frac{100}{120} \text{ of 3s.}$$
$$= 2\text{s. } 6\text{d.};$$

i.e. the mixture is actually worth 2s. 6d. per lb.

On each lb. of coffee at 3s. there is a loss of 6d.
„ „ chicory at 9d. „ gain of 1s. 9d. or 21d.
Required ratio is 21 lb. of coffee : 6 lb. of chicory ;
i.e. 7 lb. of coffee : 2 lb. of chicory.

76. EXAMPLE 1.— *A train* 180 *yd. long travelling* 20 *miles per hour passes through a station in* 45 *sec. What is the length of the station platform ?*

Total distance travelled=length of train+length of platform.

In 1 hr. the train travels 20 miles.

In 45 sec. „ „ $20 \times \frac{45}{3600} \times 1760$ yd.

$=440$ yd.

Combined length of train and platform $=440$ yd.
Length of train $=180$ „
$\therefore$ Length of platform $=260$ „

EXAMPLE 2.— *A train* 210 *yd. long travelling* 28 *miles per hour runs on a line parallel to that on which another train* 230 *yd. long is travelling at* 16 *miles per hour. How long will it take the first train to pass the second :—*

(*a*) *travelling in the same direction ?*
(*b*) „ „ *opposite directions ?*

(*a*) First train travels 28 miles per hour.
Second „ „ 16 „ „
i.e. first train travels 12 miles more than second in 1 hr ;
i.e. first train passes second train at rate of 12 miles per hour.

In passing the second train, the first must gain a distance equal to the sum of the lengths of the trains, *i.e.* a distance of 440 yd.

First train gains 12 miles in 1 hr.
„ „ 440 yd. in $1 \times \frac{1}{4} \times \frac{1}{12} \times 60$ min.
$=\frac{5}{4}$ min.

First train passes the second train in $1\frac{1}{4}$ min.

(*b*) First train travels 28 miles in one direction in 1 hr.
Second train travels 16 miles in opposite direction in 1 hr.,
$\therefore$ they pass at the rate of 44 miles per hour.

In passing, the distance covered is the sum of their lengths, *i.e.* 440 yd.

44 miles are passed in 1 hr.
$\frac{1}{4}$ mile is passed in $1 \times \frac{1}{4} \times \frac{1}{44} \times 3600$ sec.
$= \frac{225}{11}$ sec.
$= 20\frac{5}{11}$,,

i.e. the trains pass one another in $20\frac{1}{2}$ sec. (to nearest $\frac{1}{2}$ sec.).

Examples IVh

Lower.

1. A train 280 yd. long travelling at the rate of 50 miles per hour passes a second train travelling in the same direction in 45 sec. What is the speed of the second train, if the two trains are the same length ?

2. One pipe empties into a cistern 40 litres of water per second, a second empties into the cistern 80 litres per second. How long will it take to empty 8 hectolitres into the cistern ?

3. A's speed is to B's speed in the ratio 7 : 5. By how many yards can he beat B in a quarter-mile race ?

4. A five-sided figure has its interior angles 80°, 120°, 60° and of the remaining angles one is $\frac{7}{5}$ of the other. What are these two angles ?

5. Two trains pass a telegraph post; one which is 220 yd. long passes it in 9 sec. How long is the train which, going at the same rate, takes 10 sec. to pass the post ?

6. A cubic foot of water weighs 1000 oz. A solid, measuring $4\frac{1}{2}$ ft. by 4 ft. by $2\frac{1}{4}$ ft. weighs 7 ton. What is the ratio of the weight of a cubic foot of this substance to the weight of a cubic foot of water ? What is the ratio of the weight of a cubic inch of this subtance to the weight of a cubic inch of water ?

7. A man can walk 15 miles in $3\frac{1}{2}$ hours, how long will it take him to walk 24 miles if his speed is decreased by $\frac{1}{10}$th ?

8. The interior angles of a triangle are in the ratio 3 : 2 : 1. Find the angles.

9. A invests £5000 in a business, B invests £2500, and C invests £1700. C manages the business, and for this receives a salary of £400 a year out of the profits. If the total profits at the end of a year are £5000, how should this be divided among A, B, and C. ?

10. A can give B 10 yd. in a quarter-mile race, B can give C 12 yd. in a quarter-mile race. If A and C run a 400 yd. race, by how much will A beat C ?

11. A bath can be fed by the cold tap in 4 min., by the hot tap in 6 min., and can be emptied by the waste pipe in $2\frac{1}{2}$ min. How long does it take the hot and cold taps together to feed the bath ?

If the hot and cold taps are both turned on fully for 2 min. and are then turned off and the waste pipe is opened, how soon will the bath be empty ?

12. Milk at 9d. per gallon is diluted with water and then sold for 10d. a gallon at a gain of 20%. In what proportions are the milk and water mixed ?

13. B can give C 10 yd. in a race of 480 yd. How many yards could B give C in a quarter-mile race and yet win by 3 yd. ?

14. A train 320 yd. long passes a mile post in 20 sec. At what rate is the train travelling ? If this train meet another 120 yd. long travelling in the opposite direction at 45 miles per hour, how long will it take them to completely pass one another ?

15. A invests £900 in a business for 1 year, B invests £1000 for 3 months and C £1500 for 6 months. How should profits of £10,000 be divided if A receives a salary of £500 out of the profits ?

16. A clock loses 12 min. in 2 days. It is set correctly at 6 p.m. on Friday. What is the correct time when it indicates 8 p.m. on Wednesday ?

Middle.

17. A train 240 yd. long travelling at the rate of 40 miles per hour passes completely through a tunnel in $33\frac{1}{2}$ sec. Find the length of the tunnel.

18. A invests £300 in a business for 1 year, B invests £900 for 6 months, and C invests £500 for 4 months. The total profits are £790, and of this A receives £250 in salary and £100 is placed in a reserve fund. Find the money due to each of A, B, and C.

19. The hot tap can feed a bath in $7\frac{1}{2}$ min. When the cold tap is also turned on, it takes only 3 min. to fill the bath. How long does the cold tap alone take to feed it ?

20. A cistern is 8 ft. by 6 ft. by 10 ft. One pipe can carry to it 480 cu. in. of water per minute, a second carries half that amount per minute. If both these pipes be set to fill the empty cistern, how long will they take ?

Higher.

21. A can run p yd. in 1 min., B can run q yd. in 1 min. In a race A runs the whole course and wins the race in 4 min. How far is B from the winning post ?

22. A clock gains 3 mins. a day. It is set at 10 a.m. on Monday. What is the right time when it says 10 a.m. the next Monday ?

CHAPTER V

Formulæ

77. When we have to find the area of a rectangle, we multiply the number of units in its length by the number of units in its breadth.

Now the rule—

Area of rectangle = number of units of length × number of units of breadth

can be written much more simply if we take letters to represent the numbers of units of area, length, and breadth.

Let **A** = number of units of area,
" **l** = " " length.
" **b** = " " breadth,

Then the rule becomes $\mathbf{A = l \times b}$.

When the rule is written like this, not in words but in symbols, we call it the **formula** of the rule.

78. The formula in Paragraph 77 tells us *how* to obtain the area of *every* rectangle, but it does not *give* us the actual area of any *one* rectangle. To get that we must put in the formula the particular measurements of the special rectangle that we are considering.

This process is called **substituting** the given values in the formula.

All results which can be obtained from the same type of facts in the same way can be summarised in a formula.

EXAMPLE 1.—If a man's weekly expenses can be classified under board, lodging, travelling, amusements, and miscellaneous, then a formula for his savings would be :—

$$S = I - B - L - T - A - M$$

where S units represents his savings,
I " " " income, and so on.

Any one of these symbols can be found if the others are all given.

Note.—You will remember that in your arithmetic you have used 2^2 as a short way of writing 2×2, 4^5 as a short way of writing $4 \times 4 \times 4 \times 4 \times 4$, etc.

In the same way we write x^3 for $x \times x \times x$,

y^4 for $y \times y \times y \times y$—and so on.

EXAMPLE 2.—If the edge of a cube is a units, then the formula for the volume of the cube is $V = a^3$.

This is true for *every* cube, and to find the actual value of V in any special case we must substitute the special value of a.

79. You have already discovered in your Practical Geometry lessons the formulæ for the circumference and area of any circle; namely $C = 2\pi r$ and $A = \pi r^2$ where C represents the number of units of length in the circumference, r represents the number of units of length in the radius, A represents the number of units of area, and π is the ratio of the circumference of any circle to its diameter. ($\pi = \frac{22}{7}$ correct to three significant figures.)

Examples Va

Lower.

1. Using the formula $A = l \times b$, find the areas of the following rectangles :—

1. $l = 6$ in., $b = 4$ in.
2. $l = 3$ cm., $b = 1{\cdot}5$ cm.
3. $l = 25$ yd., $b = 3$ in.

2. Using the formula $A = l \times b$, find the lengths of the following rectangles :—

1. $A = 100$ sq. in., $b = 5$ in.
2. $A = 36$ sq. cm., $b = 4$ cm.
3. $A = 108$ sq. yd., $b = 9$ in.

3. Find the circumference of a circle which has a radius of 3 in.

4. Find the area of a circle which has a radius of 7 centimetres.

Middle.

5. Given that the circumference of a circle is 44 in., find the area.

6. A cart wheel makes 6 revolutions in travelling a distance of 51 yd. 1 ft. Find :—

(1) the length of its circumference,
(2) „ „ radius.

Higher.

7. The radius of a hoop is 2 ft. How many revolutions will it make when rolled along a distance of 1320 yd.?

8. Obtain the formula to give the area of the walls of any room.

80. You have already learnt how to work Simple Interest sums. We will now see if we can find any formula with which to check their answers.

Let £P represent the principal.
,, r represent the rate per cent. per annum.
,, t years represent the time.
,, £I represent the required interest.

The Simple Interest on £100 for 1 year is £r.

,, ,, ,, £P for t years is $\frac{P}{100}\times\frac{t}{1}\times£r$

$$=£\frac{Prt}{100}$$

$$\therefore \text{ our required formula is } I=\frac{Prt}{100}$$

EXAMPLE 1.—*Find the Simple Interest on* £350 *for* 5 *years at* 2%, *and check your answer by using formula.*

Simple Interest on £100 for 1 year is £2.

,, ,, £350 for 5 years is $\frac{350}{100}\times\frac{5}{1}\times£2$

$$=£35.$$

Check.—Formula is $I=\frac{Prt}{100}$.

$P=350$, $r=2$, $t=5$.

$$\therefore I=\frac{Prt}{100}=\frac{350\times2\times5}{100}=35.$$

$\therefore$ Required Interest is £35.

Examples Vb

Lower.

1. Find the formula which gives the rate per cent. at which a given principal will give a certain interest in a given time.

2. Find the formula which gives the time in which a given principal will give a certain interest at a given rate per cent.

3. In what time will £112, 10s. give an interest of £15, 15s. at 2%? Check your answer by formula.

4. At what rate per cent. will £500 give an interest of £52, 10s. in 4 years ? Check your answer by formula.

5. Given the formula $p=2(l+b)$ find :—

(1) p when $l=$ 7 in., $b=4$ in.
(2) l „ $p=30$ in., $b=5$ in.

Do you recognise this formula ?

Middle.

6. If one train travels at p miles per hour and a second train train travels at q miles per hour, find the formula which will give the speed at which the trains pass one another if p is greater than q :—

(a) When they are travelling in the same direction.
(b) „ „ „ opposite directions.

7. Find the formula for determining the exterior angle of a regular polygon. (Let n represent the number of sides.) Use the formula to find the size of the exterior angles of the following figures :—

(a) A regular pentagon.
(b) „ „ hexagon.
(c) „ „ decagon (*i.e.* a ten-sided figure).

Higher.

8. Find a formula for determining the sum of the interior angles of any polygon which has each angle less than 180°. Use this formula to find the sum of the angles of :—

(a) A pentagon.
(b) A quadrilateral.
(c) A sixteen-sided figure.
(d) A decagon.

R. B. A. N. M'S.
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CHAPTER VI

Graphs

81. The following table gives the mean temperature in degrees Fahrenheit of several places over a period of one year :—

Place.	January.	February.	March.	April.	May.	June.	July.	August.	September.	October.	November.	December.
Ben Nevis . .	24·1	23·8	23·9	27·8	33·3	39·5	40·4	39·9	37·9	30·5	28·4	24·9
Fort William .	38·6	39·0	40·4	45·1	50·1	55·2	57·1	57·5	53·6	47·4	42·7	39·3
Belmullet . .	42·7	43·0	43·8	47·4	51·0	56·1	58·1	58·4	55·7	50·1	45·7	43·2
Scarborough .	38·2	39·2	41·1	44·7	49·3	53·7	59·3	58·8	55·2	48·5	43·6	39·0
San Francisco .	50·4	51·9	53·4	54·5	56·5	58·6	58·5	58·4	59·4	59·2	55·6	51·7
Cape Town .	69·9	69·3	67·0	63·3	58·7	56·2	55·1	55·8	58·1	61·6	65·0	67·8
Brisbane . .	78·0	76·6	75·0	70·7	65·2	60·8	58·8	60·9	64·5	69·1	73·7	77·1
Calcutta . .	65·0	70·0	79·0	85·0	85·0	84·8	83·0	82·0	82·0	80·0	72·0	65·0
Peking . .	23·4	28·6	41·5	56·8	..	77·3	79·3	76·9	67·6	54·7	..	26·7
Winnipeg . .	5·2	0·7	11·6	33·8	52·4	..	..	..	..	38·3	16·7	..
New Westminster	36·8	39·3	42·7	48·6	54·2	59·7	63·1	63·6	56·9	51·0	43·1	35·6
Montreal . .	13·4	17·5	..	40·4	55·4	65·6	69·8	68·7	..	..	..	18·6

As we look at this table, we can form certain ideas about the climate of the various places mentioned. The table is not very easy to follow, there are so many figures that it is not simple to pick out any particular fact we may want to know—as, for example, which place of all those mentioned has the greatest extremes of temperature, or which place has the longest spell of very hot or very cold weather, and so on. We could find the answers to these questions from the table, but we should have to hunt through all the figures carefully and it takes much time.

It is possible to keep a record of such facts as these by a diagram or graph ; if we have the record as a graph, we shall

find it more simple to compare the climates of different places.

Suppose we begin by drawing a graph to represent the monthly temperatures for Ben Nevis. We draw a straight line and mark off along it points to represent the twelve months of the year ; we then mark a point above the name of each month at a height which will show the temperature for that month. If you try to do this, you will find that the work will be simplified if squared paper is used as it is then much easier to make small measurements quickly and accurately. When you take the squared paper, the first thing you must arrange is the size of the diagram or graph you are going to make.

Always **make the diagram as large as you can.**

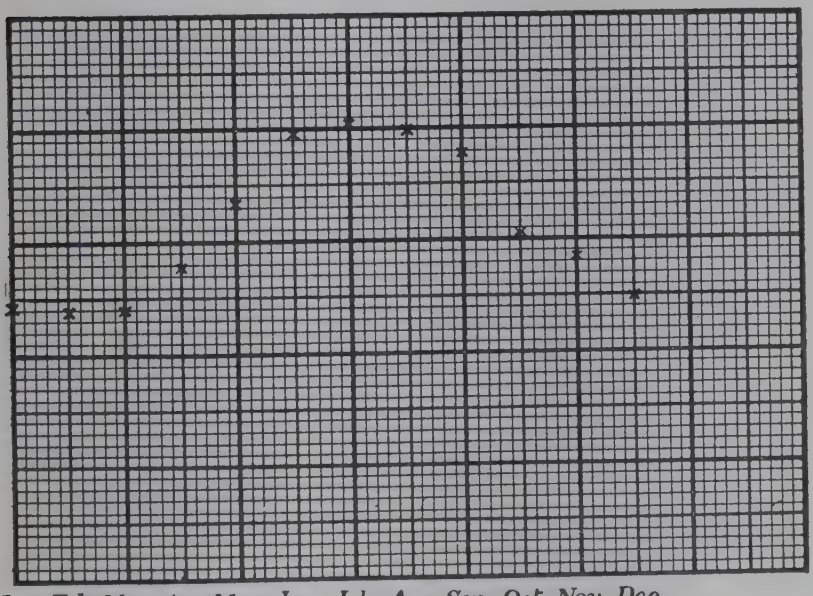


FIG. 1.

In fig. 1 we take almost the whole width of the paper to mark the months, ·5 cm. between the names of any two consecutive months ; above the line 1 mm. represents 1° Fahrenheit of temperature. When we have plotted the points as in the figure, we get a very clear idea of the temperature changes throughout the twelve months of the year.

Let us draw a similar graph to show the change in temperature at Fort William throughout the year (fig. 2).

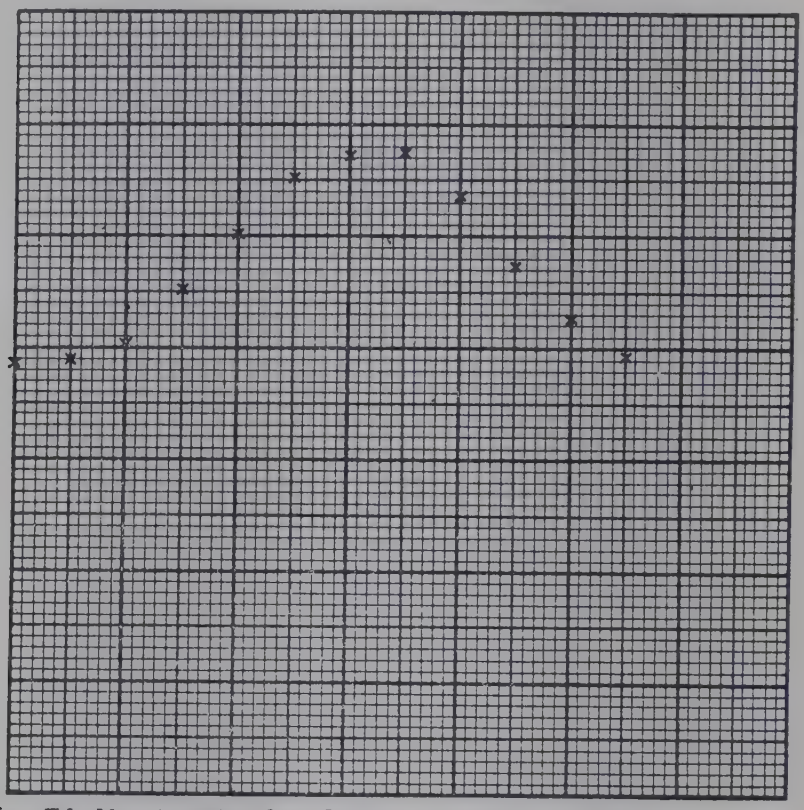


FIG. 2.

We are now able to compare with ease the temperature throughout the year of these two places which differ geographically in being—one 4000 ft. above sea-level, and the other at sea-level.

Draw the temperature charts for the other places mentioned in the table.

You will notice that in the table of values for Peking, there is no temperature given for May and November. Plot the points to represent the temperatures for the other months. Can you make any suggestion about the temperatures for May and November ?

In the table for Winnipeg there are a number of temperatures not given ; plot those that are given. In this case are you able to make suggestions about the temperature in June, July, August, September, or December ?

Consider also the missing temperatures in the table for Montreal.

Notice that, in each case where the table is complete, if we are told the month we can at once read from the graph the temperature for any given place. The mathematical way of saying this is: **"For each place, the temperature is a function of the month of the year."**

82. Five candidates A, B, C, D, and E compete for a scholarship. A gains 430, B 380, D 210, E 350 out of a possible 500 marks.

Draw a graph to show what marks each gained. Is it possible to find out from this graph anything about C's marks?

Fig. 3 shows this graph where the marks in each case are a function of the candidate.

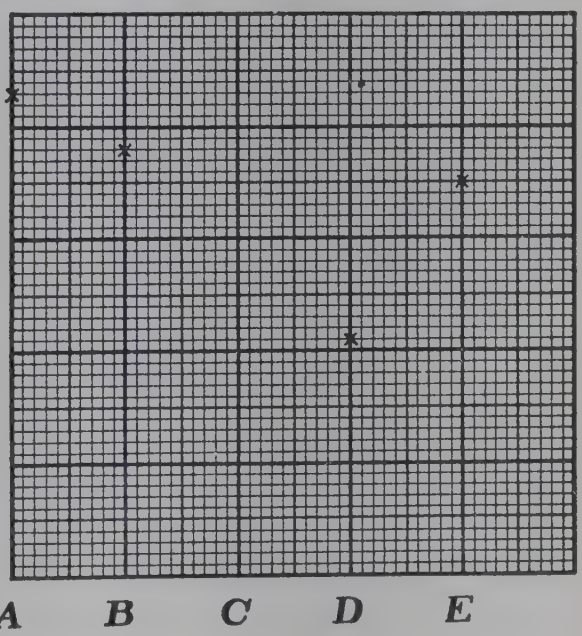


FIG. 3.

83. We see that in some graphs it is not possible to fill in a missing value. Can you explain why it is that in some cases, *e.g.* May temperature in Peking, you could fill in a fairly accurate value, whilst in others, *e.g.* August temperature in Winnipeg, your guess would be much less accurate and, in this case, about the candidate's marks, you cannot fill in the value at all? In every graph you draw you should notice whether or not it is one in which missing values can be guessed with some degree of accuracy.

84. Suppose we are told that the lengths of side of a number of squares are in order 1, 2, 3, 4, 5, 6, 7, 8 units, then the areas of the squares would be in turn 1, 4, 9, 16, 25, 36, 49, 64 square units.

We could draw a graph representing the area of a square as a function of the length of side.

Fig. 4 shows the type of graph we should obtain.

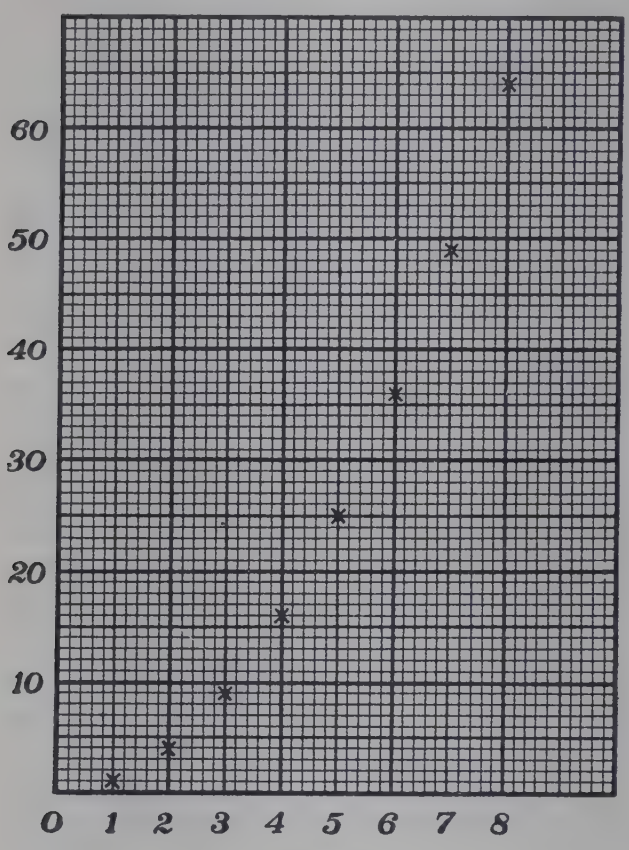


FIG. 4.

Notice that, in order that we may be able to plot a point to represent 64 square units, we have had to take a much smaller scale to represent 1 square unit than that chosen to represent 1 unit of length.

(*It is well to note this question of scale as, for every graph, we must decide what scale we shall use for each of the two sets of values we have to consider.*)

If we next plot points to represent the areas of squares of side ·5, 1·5, 2·5, 3·5, etc., units respectively and therefore of corresponding area ·25, 2·25, 6·25, 12·25, etc., square units, we find a number of points between those already plotted; we might now consider the areas of squares of side ·25, 1·25, 1·75 units respectively and so on, in fact the length of side of the square may have any value, and this could be represented along the line OX of fig. 4, and there would be a corresponding value above the line OX to represent the area of each square. In a case like this, where **each point along the line can be a value of which the graph is a function, we join the various points of the graph by a smooth curve.**

85. EXAMPLE 1.—Draw a graph to represent the area of a circle as a function of the radius.

EXAMPLE 2.—Draw a graph to represent the volume of a cube as a function of the length of an edge for the following values of the edge: ·5, 1, 1·5, 2, 2·5, 3, 3·5, 4 units.

EXAMPLE 3.—Draw a graph to represent the Simple Interest on £100 at 5% as a function of time for periods: 1, 2, 3, 4, 5, 6 years.

EXAMPLE 4.—Draw a graph to represent the amount of £100 at 4% Simple Interest as a function of the time for periods of 1 to 6 years.

EXAMPLE 5.—Draw a graph to represent the distance travelled by a train going 45 miles per hour as a function of the time for a period 1 to 15 years.

EXAMPLE 6.—Draw a graph to represent pounds sterling as a function of francs when the rate of exchange is 180 francs to £1.

86. We have seen that when two quantities are so related that the value of one must be considered if the value of the other is to be found, then the second is said to be a function of the first.

For example, in fig. 1, if we look at the month January, our choice of month determines the mean temperature; if we alter to the month June, we get another value for the temperature; therefore the mean temperature is said to be a function of the month of the year.

In fig. 3, if we consider candidate A, we see that the marks gained are 430; if we change to candidate B, the marks also are different, 380; therefore the number of marks is said to be a function of the candidate.

In the same way we can consider any expression containing x as *a function of* x, because the value of the expression will be determined by the value we give to x.

87. We shall now draw the graphs of certain simple functions of x. To avoid writing the phrase **function of** x, it is usual to write the shortened form **f(x)**.

We shall first consider the graph of $f(x)=x^2$.

Take values 1, 2, 3, 4, 5 for x, and mark off along a horizontal line on squared paper points to represent these values.

Corresponding to the value 1 for x, is the value 1 for $f(x)$;
" " 2 for x, " " 4 for $f(x)$;
" " 3 for x, " " 9 for $f(x)$;
etc.

Mark, therefore, points 1, 4, 9, etc., units respectively above the points 1, 2, 3, etc., along the horizontal line.

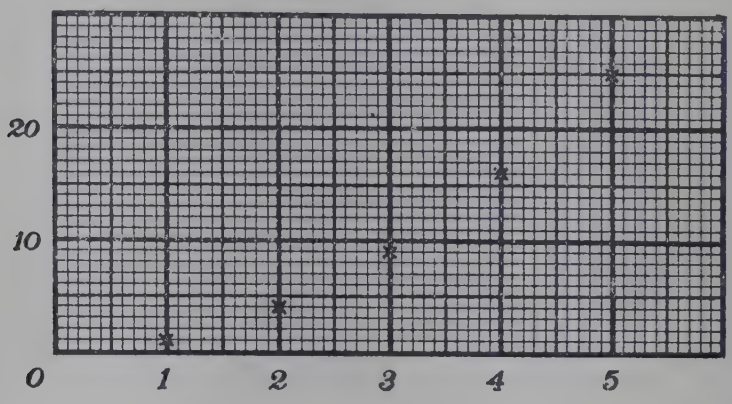


FIG. 5.

Do these points appear to lie on a curve ?

Do the facts make it reasonable to suppose that the line joining the points will have a meaning between the values taken ? *E.g.* could x have the values 1·5, 2·75, 3·2, etc. ? If so, what would be the corresponding values for $f(x)$?

Draw the curve and check the values of $f(x)$ corresponding to 1·5, 2·75, 3·2 for x.

What value has the function when $x=0$?

88. In Chapter II we talked about negative numbers, and we saw that from one point of view the negative sign attached to a number could be understood to give a meaning exactly opposite to the meaning understood to belong to a positive number—*e.g.* if, when we take $x=4$, we mark off along the horizontal line OX a distance 2 in. to the right of O, then, when $x=-4$, we should mark off a point 2 in. to the left of O. In the same way if, when $f(x)$ has a positive value, we measure a distance above the horizontal, then, when $f(x)$ has a negative value, we should measure a distance below the horizontal line.

With this in mind, let us try to plot points in the graph of $f(x)=x^2$ when x has negative values.

What sign have the values of $f(x)$?

Will the points representing $f(x)$, when x has the values -1, -2, -3, -4, be above or below the horizontal line ?

89. Now consider the values of $f(x)$ if $f(x)=x^3$. First plot points when $x=\cdot5$, 1, 2, 3, 4. In what way does the graph of this function (*a*) resemble, (*b*) differ from that of $f(x)=x^2$?

Take negative values for x, let $x=-\cdot5$, -1, -2, -3, -4. Will the points on the graph corresponding to these values of x be above or below the horizontal line ?

Plot on one piece of squared paper the values of $f(x)=x^3$ when $x=-4$, -3, -2, -1, 1, 2, 3, 4.

Is it reasonable to join the points you plot by a curve ? Can you guess anything about the part of the curve when x is between -1 and $+1$ in value ? Draw in this part of the curve with a dotted line. To verify the correctness of your guess, you might work out all the values of x^3 when x is $\pm\cdot5$, $\pm\cdot4$, $\pm\cdot3$, $\pm\cdot2$, $\pm\cdot1$, etc., and compare the results with the values you can read from your own graph.

Is it easy to see these points on the graph ? As you try to read these values, you will easily understand how much better it would be to copy the central part of the graph on a larger scale.

Take a fresh piece of squared paper and let 6 in. represent the difference between $+1$ and -1 along the horizontal line. Also take 6 in. to represent the difference between $+1$ and -1 for values of $f(x)$. With this scale plot the points where $x=\pm1$, $\pm\cdot7$, $\pm\cdot6$, $\pm\cdot5$, $\pm\cdot4$, $\pm\cdot2$, 0. Compare the curve you get by joining these points with that which you "*guessed*" on the graph of smaller scale.

Draw graphs of : $f(x)=x+7$,
$f(x)=x-3$,
$f(x)=(x+7)(x-3)$.

90. We shall next consider the graph of $f(x)=\frac{1}{x}$ from $x=1$ to $x=6$.

Can we make any general remark about the size of $f(x)$ here as compared with the size of x ? If we go on increasing the size of x, what happens to $f(x)$? If, on the other hand, we take smaller and smaller values for x, what happens to the corresponding values for $f(x)$?

Take a piece of squared paper and draw a horizontal line across the middle of it. Mark values for x from -10 to $+10$ along this line.

Draw a line perpendicular to the horizontal line through the point where $x=0$ is marked.

Plot the points on the graph $f(x)=\frac{1}{x}$ for values of x from $\frac{1}{10}$ to 10. (Use the same scale for $f(x)$ as for x.) Plot the points on the graph corresponding to $x=-4, -2, -1, -\cdot5, -\cdot25$; from these draw as accurately as you can the graph of $f(x)=\frac{1}{x}$ for values of x between -10 and $-\frac{1}{10}$.

91. Every graph is the record of two sets of values, one set being a function of the other.

The word **variable** is used to refer to a whole set of values of different magnitudes but all of the same type—*e.g.* in the graph where the volume of a cube is a function of the edge, one variable is the edge having the different values 1, 2, 3, etc., units, and the other variable is the volume with the different values, 1, 8, 27, etc., cubic units.

Examples VIa

1. Draw graphs to represent the sun and shade temperatures as a function of the time of day for the 24th and 25th May given the following readings.

Summer Time.	24th May.		25th May.	
	Sun.	Shade.	Sun.	Shade.
10 a.m.	87	60	93	62
11 a.m.	93	63	93	67
12 noon	107	66	110	67
1 p.m.	109	68	92	67
2 p.m.	101	68	94	69
3 p.m.	100	69	88	69
3·30 p.m.	105	69	83	70

2. Draw two graphs on the same paper: (*a*) to represent the ordinary 3rd class return fare from London, (*b*) to represent the 3rd class week-end return fare from London as a function of the place for the following resorts :—

Place.	Ordinary 3rd Class return fare.	Week-end fare, 3rd Class.
Bournemouth	27s. 0d.	18s. 0d.
Bognor	16s. 8d.	11s. 3d.
Eastbourne	15s. 0d.	10s. 0d.
Hayling Island	17s. 10d.	12s. 0d.
Littlehampton	15s. 4d.	10s. 3d.
Weymouth	35s. 10d.	24s. 0d.

3. (*a*) From the graphs drawn in answer to Question 2, can you make any general statement about the week-end fares in comparison to the ordinary 3rd class return fares ?

(*b*) Can you deduce (i) accurately, (ii) approximately, the week-end fare to the following places for which the ordinary 3rd class return fares are given :—

Exeter, 43s. ; Folkestone, 17s. 10d. ; Hastings, 15s. 2d. ; Ilfracombe, 50s. 6d. ; Lyme Regis, 38s. 0d. ?

(*c*) The week-end fares to Portsmouth and Whitstable are respectively 12s. 6d. and 10s. Find from the graphs of Exercise 2, the return fare ordinary 3rd-class rate. State in each case whether the answer is an approximation.

4. Draw graphs of the following functions of x :—

(i) $f(x)=(x+4)(x+2)$. (iv) $f(x)=(x-2)(x+3)$.
(ii) $f(x)=x^2+2x$. (v) $f(x)=2-3x$.
(iii) $f(x)=5x+7$. (vi) $f(x)=3x^2-7x+1$.

(vii) $f(x)=\frac{1}{2x+1}$ (*between* $x=1$, $x=5$).

(viii) $f(x)=x^3-4x$.

(ix) $f(x)=\frac{1}{x+3}$ (*between* $x=-2$, $x=4$).

(x) $f(x)=2x-3x^2$.

92. You will have noticed that we have drawn graphs of different types : (*a*) those in which we can plot separate points only, and where knowledge of a set of values does not give us information about values not mentioned, *e.g.* candidates and marks ; (*b*) those in which, given a set of values, intermediate values can be inserted with a certain degree of accuracy ; (*c*) those graphs which may be described as *regular* or *smooth*.

Notice that in each case where we have drawn a graph of $f(x)$=an algebraic expression containing x, we have found the graph to be of the last of the three types. Among the *regular* graphs, we have drawn many that are curved and some that are straight lines.

93. We shall now try to find out what kind of algebraic expression will give a straight line graph.

Look back at the problems from which you have obtained straight line graphs :—

(*a*) Simple Interest . . . time graph.
(*b*) Amount at Simple Interest . „ „
(*c*) Distance . . . . „ „
(*d*) Francs . . . . pounds sterling graph.

Graphs (*a*), (*c*), (*d*) are straight lines starting from the point where one variable is zero and the other variable is also zero.

Graph (*b*) is a straight line starting from a point where one variable is zero but the other variable is **not** zero.

We have already considered the question of Simple Interest (*a*) in our work on ratio, (*b*) in our work on formulæ; we have found that Simple Interest varies directly as time, and that this can be expressed as a ratio :—

$$\frac{\text{Simple Interest}}{\text{time}} = \text{rate},$$

or using the formula $\frac{I}{t} = r$

or $I = rt$,

where r is always the same for any one problem.

In the same way we can obtain a formula to represent the distance travelled in any time by a train travelling at a given rate : $D = RT$.

Or we can write down a formula to express the number of francs equivalent to a sum in pounds sterling on any given day. $f = a$ £ where a gives the rate of exchange for £1 on the given day.

These three formulæ, themselves general statements, can be grouped under one expression $\mathbf{f(x) = mx}$; **the most general way of writing that one quantity varies directly as another.**

94. The graph of $f(x)=mx$ is a straight line. Instead of $f(x)=mx$, we sometimes write $y=mx$; the advantage of the first method is that it particularly reminds us that one variable is a function of the other; the advantage of the second method is that it particularly reminds us that we are dealing with **two** variables.

We obtained formulæ for three of the four graphs quoted above but not for graph (*b*), the amount at Simple Interest —time graph. The amount in any given problem is made up of two parts: (*a*) the Simple Interest which varies directly as the time, and (*b*) the Principal which remains constant: this can be written as a formula—

$$A=rt+P,$$

or quite generally $$f(x)=mx+c,$$

or $$y=mx+c.$$

The graph of this expression is a straight line and "c" is the value of the function when the variable x is zero.

95. Let us draw the graph of the line represented by $y=3x+5$.

Here x, and $f(x)$ or y can both be either positive or negative, so we shall need negative as well as positive values along the horizontal line (*or the* **x axis** *as it is called*); and negative values below the x axis as well as positive values above it for $f(x)$.

(*The line through the point where* $x=0$, *perpendicular to the x axis, is called the* **y axis**: *the x and y axes cut at a point called* **the origin.**)

As the graph will be a straight line, only two points need be plotted to get the direction of the line; it is usual, however, to plot three points; the third point serves as a check on the other two.

We choose three values for x and calculate the corresponding values for $f(x)$ or y, *e.g.*—

if $x=-2, 0, 2,$

then $f(x)$ or $y=-1, 5, 11,$

We plot the points determined in the above table and get the graph (fig. 6).

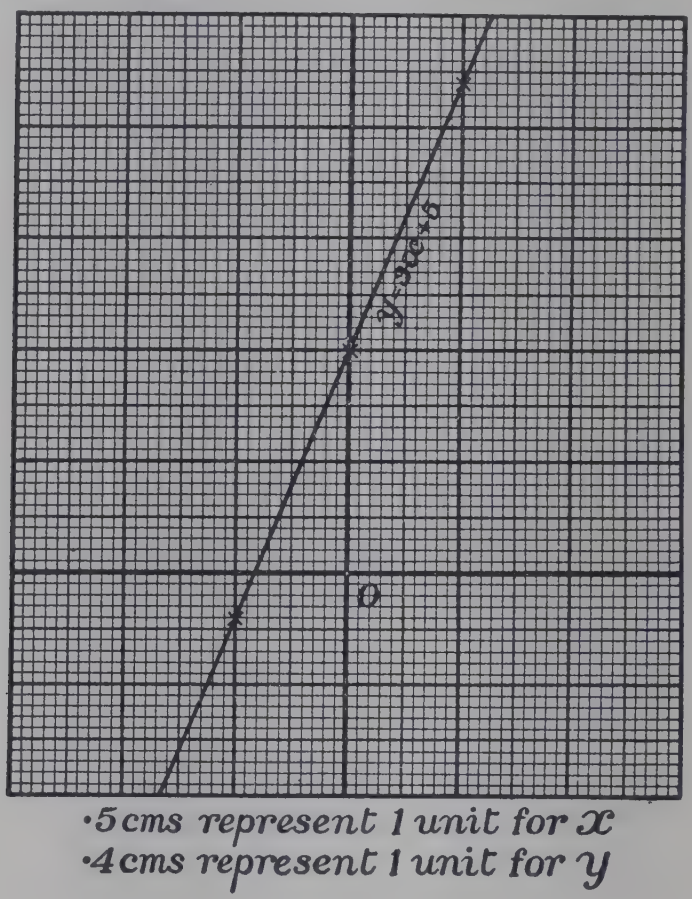


FIG. 6.

Each pair of values of x and y, *e.g.* $\begin{cases} x=2 \\ y=11 \end{cases}$ or $\begin{cases} x=4 \\ y=17 \end{cases}$

etc., represents a point on the graph of $y=3x+5$. This line is quite unlimited in length; we are able to show only a portion of it on the squared paper, but it contains an unlimited number of points. When you are learning how to solve simultaneous equations, you learn that if you have **one** *equation containing two unknown quantities, there are an unlimited number* of pairs of values which satisfy the equation. These solutions correspond to the unlimited number of points on the line represented by the equation.

96. Draw on the same piece of squared paper and with the same scale the graph represented by the equation (1) $y=3x+5$, and the graph represented by the equation (2) $3y=5-x$.

The table of values for x and y in (1) has already been worked out ; the table for (2) is as follows :—

When

	$x=2,$	$0,$	5
$f(x)$ or $y=$	$1,$	$1\frac{2}{3},$	$0.$

Each point on the graph representing (1) gives one of the possible solutions (pair of values for x and y) which satisfy $y=3x+5$; each point on the graph representing (2) gives one of the possible solutions which satisfy $3y=5-x$.

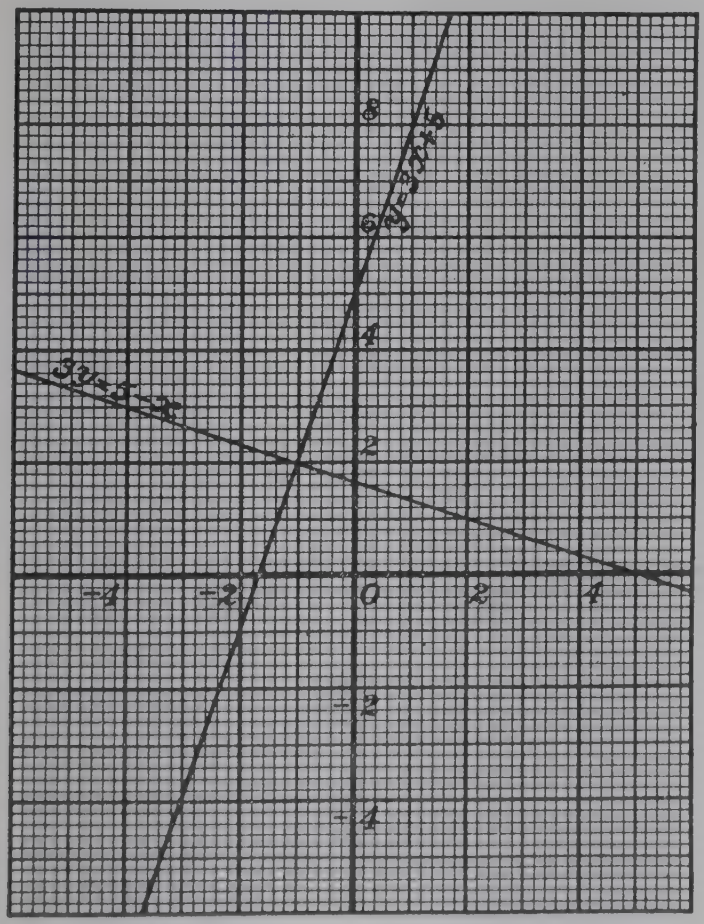


Fig. 7.

Notice that if $\begin{cases} y=3x+5 \\ 3y=\ 5-x \end{cases}$ are regarded as simultaneous equations, **the one set of values** for x and y which **satisfies both equations** represents the **one point** which lies on both lines, *i.e.* the point where the lines cut. From your own graphs, what is this point ?

Verify by solving the two equations.

97. It is clear that we can solve simultaneous equations by means of graphs.

Examples VIb

By means of graphs solve the following simultaneous equations :—

1. $\begin{cases} y=3x-4. \\ 2y=\ x+2 \end{cases}$

2. $\begin{cases} 2x+\ y=5 \\ 3x-4y=2. \end{cases}$

3. $\begin{cases} x=y+3 \\ 2x=y+10. \end{cases}$

4. $\begin{cases} y\ =3x-1 \\ y-x=-3. \end{cases}$

98. Do you consider that this method of solving simultaneous equations has any special advantages or disadvantages ? If so, what are they ? Do two straight lines, if produced to any length, always cut ?

We know from our work in geometry that, if two straight lines are parallel, they do not cut. It is interesting to see if there is any method of finding out from two equations, each representing a straight line, whether the two lines cut or are parallel.

We have already drawn the graphs of two parallel straight lines : (i) the graph showing Simple Interest as a function of time, and (ii) the graph showing amount at Simple Interest as a function of time.

We found also that the equations representing these two graphs could be written in the general form (i) $y=mx$, (ii) $y=mx+c$ where " m " represents the interest per annum on the principal and " c " the principal.

FIG. 8.

Fig. 8 shows the Simple Interest and the Amount of £100 for 1, 2, 3, 4, 5, 6 years at 1 %.

99. Draw graphs to represent Simple Interest as a function of time, and amount at Simple Interest as a function of time for rates $2\frac{1}{2}\%$, 5%, $4\frac{1}{2}\%$, and for principals £100, £200, £500. You will find that, in the cases where the interest per annum is the same, the graphs are parallel lines, *i.e.* with the same value for "m" the lines remain parallel whatever value "c" may have.

100. Draw the graphs of the lines represented by

$$(a)\ y=4x+5, \quad (b)\ y=4x-2, \quad (c)\ 2y=8x-9.$$

These three lines are parallel; they cut the x axis at the same angle (*i.e.* corresponding angles are equal; Assumption (4 (*b*)). Each one of these lines cuts the y axis at a different point. Do you notice anything about the distance from

the origin of the points where the graphs cut the axis ? We now see that apparently the direction of the line represented by the equation $y=mx+c$ depends on "m," and the position of the line on "c."

Test this for several cases, *e.g.* draw the graphs represented by

(*a*) $y=5x+4$. (*c*) $y=3x+4$.
(*b*) $y=5x-2$. (*d*) $y=3x-2$.

101. So far we have not proved that the direction of the line depends on "m" and the position on "c," we have only concluded that it is so by considering the graphs we have drawn ; later on you will learn to prove this, for the present we must take it as an assumption.

Examples VIc

1. Draw the graph which represents $f(x)=x^3-2x$ between values $x=4$, $x=-4$.

2. Draw a graph to represent the runs as a function of the batsman for the following cricket innings :—

SURREY.

Hobbs, c. Hardinge, b. Woolley	77
Sandham, c. Hubble, b. Woolley	49
A. Jeacocke, c. Legge, b. Hardinge	76
Shepherd, b. Woolley	28
D. R. Jardine, c. Hubble, b. Ashdown	23
Ducat, c. Seymour, b. Collins	44
E. R. T. Holmes, not out	62
P. G. H. Fender, b. Wright	42
Peach, run out	22
Strudwick, not out	1
B. 6, l.b. 4	10
(Total, 8 wkts.)	434

Fenley to bat.

3. Make in the form of a graph a record of the maximum value in French francs of the pound sterling each day for a week.

4. Draw a graph to represent $f(x)=3x+1$. Does this cut the graph which represents $f(x)=3x-4$? Can you make any statement about the two equations

(i) $y=3x+1$, (ii) $y=3x-4$?

5. Draw the graph to represent $y=4x+1$. On the same paper draw a graph similar to this one and parallel to it but one unit above it; can you suggest the equation that this graph represents?

6. Draw the graph which represents $f(x)=2-3x$. Can you suggest the direction and position of the graph which represents $f(x)=3x-2$? Draw the latter graph and verify the correctness of your opinion about it.

7. Keep a record of the temperature of your form room throughout one week. Take readings before and after morning school. Explain how these readings might be represented in graphical form. Note particularly such points as the advisability of keeping both sets of readings in the same graph, and the type of graph obtained (connected or unconnected points).

8. Draw the graph to represent $f(x)=\frac{1}{x^2}$ between $x=\frac{1}{4}$ and $x=4$, and between $x=-\frac{1}{4}$ and $x=-4$.

9. Draw the graph to represent $f(x)=(x-5)(2x+1)$.

10. Solve graphically $\begin{cases} \text{(i)} & y=3x+7. \\ \text{(ii)} & x+4y=2. \end{cases}$

CHAPTER VII

Factors

102. Suppose that I have 5 sixpences and 4 sixpences. Then I may say that I have (5+4) times sixpence or (5+4) sixpences.

Now consider the expression $ba+ca$.

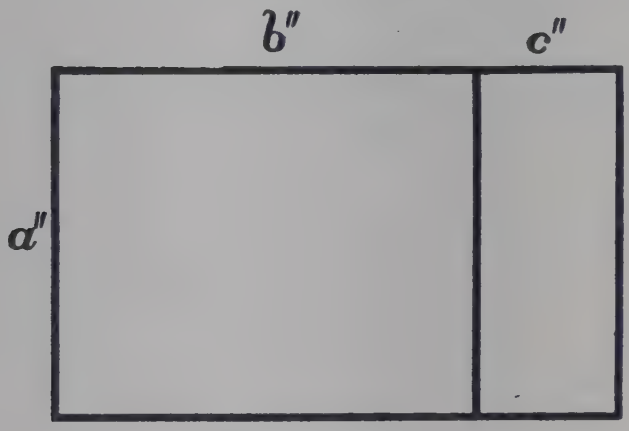
This may be written (b times a)+(c times a),

or ($b+c$) times a;

i.e. $(b+c)a$ or $a(b+c)$,

$\therefore\ ba+ca=a(b+c)$.

This can be seen from the following illustration.



Suppose we have a piece of paper of sides a in. and $(b+c)$ in. divided by a straight line, parallel to the side a in., which cuts the other side into the parts b in. and c in., as in the diagram. Then the areas of the 2 parts of the paper are ba sq. in. and ca sq. in. These 2 areas together must equal the total area of the paper, namely $a(b+c)$ sq. in.,

$$\therefore\ ba+ca=a(b+c).$$

Since a multiplied by $(b+c)$ gives $ba+ca$, we may say that the **factors** of $ba+ca$ are a and $b+c$.

EXAMPLE 1.—*Factorise* $3ax+6ay$.

$$\begin{aligned} 3ax+6ay &= x \text{ times } 3a+2y \text{ times } 3a \\ &= (x+2y) \text{ times } 3a \\ &= 3a(x+2y). \end{aligned}$$

EXAMPLE 2.—*Factorise* $ab+ac+ad$.

$$\begin{aligned} ab+ac+ad &= b \text{ times } a+c \text{ times } a+d \text{ times } a \\ &= (b+c+d) \text{ times } a \\ &= a(b+c+d). \end{aligned}$$

EXAMPLE 3.—*Factorise* $10xy-15xa+25xz$.

$$\begin{aligned} 10xy-15xa+25xz &= 2y \text{ times } 5x-3a \text{ times } 5x+5z \text{ times } 5x \\ &= (2y-3a+5z) \text{ times } 5x \\ &= 5x(2y-3a+5z). \end{aligned}$$

Examples VIIa

Find the factors of—

Lower.

1. $xa+xb$.
2. $3bc+6bd$.
3. $4ax-2bx$.
4. $3abc-4abd$.
5. $15abx-5adx+10bcx$.
6. $14ac+21cd$.
7. $12ax-18ay$.
8. $10abc+18bcd-24bcx$.

Middle.

9. $12ax-7bxy-14cx+24xy$.
10. $\frac{2}{3}ab+\frac{4}{3}ac$.

Higher.

11. $\frac{1}{5}by+\frac{3}{5}bx-\frac{4}{5}ba+\frac{2}{5}b$.
12. $x+xy+xz$.

103. EXAMPLE 1.—*Now consider the expression*

$$ax+bx+ay+by.$$

$$\begin{aligned} ax+bx+ay+by &= a \text{ times } x+b \text{ times } x+a \text{ times } y+b \text{ times } y \\ &= x(a+b)+y(a+b) \\ &= x \text{ times } (a+b)+y \text{ times } (a+b) \\ &= (x+y)(a+b). \end{aligned}$$

That is, the factors of $ax+bx+ay+by$ are $(x+y)$ and $(a+b)$.

This result is easy to check.

When we multiply $(x+y)$ by $(a+b)$ the result should be $ax+bx+ay+by$.

$$\begin{aligned} (x+y)(a+b) &= x(a+b)+y(a+b) \\ &= ax+bx+ay+by. \end{aligned}$$

EXAMPLE 2.—*Factorise* $ac-bc+ad-bd$.

$$ac-bc+ad-bd=c(a-b)+d(a-b)$$
$$=(c+d)(a-b).$$

Check : $(c+d)(a-b)=c(a-b)+d(a-b)$
$$=ca-cb+da-db.$$

EXAMPLE 3.—*Factorise* $4ay-6az+10cy-15cz$.

$$4ay-6az+10cy-15cz=2a(2y-3z)+5c(2y-3z)$$
$$=(2a+5c)(2y-3z).$$

Check : $(2a+5c)(2y-3z)=2a(2y-3z)+5c(2y-3z)$
$$=4ay-6az+10cy-15cz.$$

EXAMPLE 4.—*Factorise* $5ax-5bx-2ay+2by$.

$5ax-5bx-2ay+2by=5x(a-b)-2y$ times $a-2y$ times $(-b)$
$$=5x(a-b)-2y(a-b)$$
$$=(5x-2y)(a-b).$$

Check : $(5x-2y)(a-b)=5x(a-b)-2y(a-b)$
$$=5ax-5bx-2ay+2by.$$

EXAMPLE 5.—*Factorise* $ax-ay-bx+by$.

$$ax-ay-bx+by=a(x-y)-b(x-y)$$
$$=(a-b)(x-y).$$

EXAMPLE 6.—*Factorise* x^2+2ax.

Remember that x^2 means $x \times x$, just as in arithmetic 4^2 means 4×4.

$$x^2+2ax=x(x+2a).$$

Examples VIIb

Find the factors of :—

Lower.

1. $2ax+2ay+3bx+3by$.
2. $cx+dx+cy+dy$.
3. $3ax+3bx+5ay+5by$.
4. $2ac-2ad-7bc+7bd$.
5. $20ax+5ab-4cx-bc$.
6. $12ax-3ab-8cx+2bc$.
7. ax^2+abx.
8. $4x^3+2x^2-6x$.
9. $3x^2+6x+7ax+14a$.
10. $ax^2-4ax+bx-4b$.

Middle.

11. x^3+x^2+6x+6.
12. $4ax^3-4x^2-ax^2+x$.
13. $ax^2+bx+ayx+by$.
14. $a^2x^2-2a^2xy+2yc-cx$.

Higher.

15. $3x^3+21x-8x^2-56$.
16. $8ax-28x-6ay+21y$.

104. EXAMPLE 1.—*Now consider* $(x+2)(x+3)$.

$$\begin{aligned}(x+2)(x+3)&=x(x+3)+2(x+3)\\&=1x^2+3x+2x+6\\&=x^2+5x+6.\end{aligned}$$

Notice from the second line

$$3+2=5$$
$$3\times2=6\times1.$$

EXAMPLE 2.—$(x+2)(x-3)=x(x-3)+2(x-3)$

$$=1x^2-3x+2x-6$$
$$=x^2-x-6.$$

Notice again

$$-3+2=-1$$
$$-3\times2=-6\times1.$$

EXAMPLE 3.—$(x+4)(x-1)=x(x-1)+4(x-1)$

$$=1x^2-x+4x-4$$
$$=x^2+3x-4.$$

Again

$$-1+4=3$$
$$-1\times4=-4.$$

EXAMPLE 4.—$(x-6)(x-2)=x(x-2)-6(x-2)$

$$=1x^2-2x-6x+12$$
$$=x^2-8x+12.$$

Again

$$-2\quad-6=-8$$
$$-2\times-6=12.$$

If we are given an expression of the type $x^2+8x+12$ to factorise, we first write it in the form $x^2+6x+2x+12$ (since $6+2=8$, $6\times2=12$);

i.e. $x(x+6)+2(x+6)$ (1)

or $(x+2)(x+6)$.

Often you will be able to omit line (1) because you can see at sight what are the factors of the given expression.

EXAMPLE 5.—*Factorise* $x^2+4x-45$.

We can write this in the form $x^2+9x-5x-45$
since $9-5=4$
$9\times-5=-45$,
i.e. $x(x+9)-5(x+9)$
or $(x-5)(x+9)$.

EXAMPLE 6.—*Factorise* $x^2-7x+12$.

$$x^2-7x+12=x^2-?x-?x+12.$$

(Find the two $-^{ve}$ numbers which, added together, give -7 and multiplied together give 12.)

Then $x^2-7x+12=x^2-?x-?x+12$
$=x(x-?)-?(x-?)$
$=(x-?)(x-?)$.

105. *Multiply* $(2x+5)$ *by* $(3x+4)$ . . . . . (i)

Let us write this multiplication out in full, keeping our numbers in factor form.

$$\begin{aligned}&(2x+5)\\ &\times\ (3x+4)\\ =&(2\times3)x^2+\left.\begin{matrix}(3\times5)\\ +(2\times4)\end{matrix}\right\}x\ +(5\times4)\\ =&6x^2+23x+20 \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}\end{aligned}$$

Notice how the factors of the first and last terms of the two given expressions are grouped together in order that the middle term of the product may be obtained.

So the middle term of $(3x+4)$ multiplied by $(5x+2)$ will be $(3\times2)x+4\times5\ (x)$,

i.e. $36x$.

Therefore, if we are asked to factorise $35x^2+31x+6$, we must find two factors of (35×6) whose sum is 31.

The required factors are 21 and 10.

$$\begin{aligned}\therefore\ 35x^2+31x+6&=35x^2+21x+10x+6,\\ &=7x(5x+3)+2(5x+3),\\ &=(7x+2)(5x+3).\end{aligned}$$

EXAMPLE 1.—*Can* $7x^2+17x+6$ *be divided up into factors in this way?*

If it can be so divided, we must be able to arrange it so that the factors of 7×6, *i.e.* of 42, can be grouped to give by addition the number 17.

If we grouped the factors 14 and 3, we should have what we want;

i.e.

$$\begin{aligned}&7x^2+17x+6\\&=7x^2+14x+3x+6\\&=7x(x+2)+3(x+2)\\&=(7x+3)(x+2).\end{aligned}$$

EXAMPLE 2.—*Can factors of* $3x^2+2x-8$ *be found iu this way?*

The product of the end factors 3 and -8 is -24.
Are there any factors of -24 whose sum is $+2$?
Yes. The required factors are $+6$ and -4.

$$\begin{aligned}&3x^2+2x-8\\&=3x^2+6x-4x-8\\&=3x(x+2)-4(x+2)\\&=(3x-4)(x+2).\end{aligned}$$

EXAMPLE 3.—*Find the factors of* $20x^2-x-21$.

$$\begin{aligned}&20x^2-x-21\\&=20x^2+20x-21x-21\\&=20x(x+1)-21(x+1)\\&=(20x-21)(x+1).\end{aligned}$$

EXAMPLE 4.—*Find the factors of* $6x^2+21x+18$.

"Take out" the common factor first.

$$\begin{aligned}&6x^2+21x+18\\&=3(2x^2+7x+6)\\&=3(2x^2+4x+3x+6)\\&=3\{2x(x+2)+3(x+2)\}\\&=3(2x+3)(x+2).\end{aligned}$$

Examples VIIc

Find the values of the following products :—

1. $(x+3)(x+4)$.
2. $(x+7)(x+3)$.
3. $(x+2)(x+8)$.
4. $(x+5)(x+9)$.
5. $(x-8)(x+2)$.
6. $(x+8)(x-2)$.
7. $(x-4)(x+7)$.
8. $(x+4)(x-7)$.
9. $(x-11)(x+3)$.
10. $(x+5)(x+12)$.
11. $(x-8)(x-5)$.
12. $(x-7)(x-3)$.
13. $(2x+4)(x+3)$.
14. $(2x+5)(x+7)$.
15. $(3x+2)(2x+1)$.
16. $(4x-3)(2x+1)$.
17. $(3x+2)(5x-1)$.
18. $(5x-2)(x-3)$.
19. $(2x-7)(3x-2)$.
20. $(2x+3)(2x-3)$.
21. $(3x+1)(3x-1)$.
22. $(4x-3)(3x+5)$.
23. $(2x+9)(3x+10)$.
24. $(x+8)(4x-5)$.
25. $(3x-7)(2x-7)$.

Examples VIId

Find the factors of the following, and check results by multiplication :—

Lower.

1. x^2+2x+1.
2. x^2+3x+2.
3. x^2-2x-8.
4. x^2+2x-8.
5. $x^2+12x+35$.
6. $x^2+2x-48$.
7. $x^2+4x-45$.
8. $3x^2+22x+7$.
9. $8x^2-6x-5$.
10. $6x^2+13x+6$.
11. $6x^2-5x-6$.
12. $21x^2-22x-8$.
13. $4x^2-x-18$.
14. $x^2-5x-50$.
15. $12x^2+4x-1$.
16. $3x^2+2x-1$.

Middle.

17. $20x^2+31x-7$.
18. $24x^2-34x-10$.
19. $9x^2-38x+8$.
20. $15x^2+7x-2$.

Higher.

21. $54x^2+69x+7$.
22. $14x^2+15x-11$.
23. $28x^2+29x+6$.
24. $28x^2-x-2$.
25. $9x^2-10x+1$.

106. EXAMPLE 1.—*Find the factors of* x^2-81.

$$\begin{aligned} x^2-81 &= x^2+9x-9x-81 \\ &= x(x+9)-9(x+9) \\ &= (x-9)(x+9). \end{aligned}$$

EXAMPLE 2.—*Find the factors of* $16x^2-9$.

$$\begin{aligned}16x^2-9&=16x^2+12x-12x-9\\&=4x(4x+3)-3(4x+3)\\&=(4x-3)(4x+3).\end{aligned}$$

EXAMPLE 3.—*Find the factors of* $25x^2-121$.

$$\begin{aligned}25x^2-121&=25x^2+55x-55x-121\\&=5x(5x+11)-11(5x+11)\\&=(5x-11)(5x+11).\end{aligned}$$

Notice, from the above examples, that the difference between any two squares can always be expanded into the product of the sum and the difference of the roots of these squares.

EXAMPLE 4. $100a^2-36$.

$$=(10a+6)(10a-6).$$

EXAMPLE 5. $225x^2-49y^2$.

$$=(15x+7y)(15x-7y).$$

Just as in arithmetic :—

EXAMPLE 6. 37^2-34^2

$$\begin{aligned}&=(37+34)(37-34)\\&=71\times3\\&=213.\end{aligned}$$

Examples VIIe

Lower.

Factorise :—

1. x^2-16.
2. $4x^2-9y^2$.
3. $16x^2-25a^2$.
4. $25x^2-1$.
5. $36-y^2$.
6. $81-4x^2$.
7. $1-9a^2$.

Find the value of :—

8. 39^2-9^2.
9. 101^2-95^2.
10. 27^2-24^2.

Middle.

Factorise :—

11. $9a^4-a^2x^2$.
12. $81a^4-b^2$.
13. a^4-b^4.

Higher.

Factorise :—

14. a^2b^2-1.
15. $4a^2x^2-\frac{25}{a^2}$.
16. $a^4b^4-\frac{1}{c^2}$.

107. We are often able to simplify fractions by factorising the numerator and denominator.

EXAMPLE 1.—*Simplify* $\frac{x^2+7x+12}{x^2+8x+15}$.

Factorising numerator and denominator, we have—

$$\frac{x^2+7x+12}{x^2+8x+15}=\frac{(x+3)(x+4)}{(x+3)(x+5)}.$$

Since the value of any fraction is unaltered if both numerator and denominator are divided by the same expression, we have—

$$\frac{x^2+7x+12}{x^2+8x+15}=\frac{(x+3)(x+4)}{(x+3)(x+5)}=\frac{x+4}{x+5}.$$

EXAMPLE 2.—*Simplify* $\frac{x^2+4x}{x^2+5x}$.

$$\frac{x^2+4x}{x^2+5x}=\frac{x(x+4)}{x(x+5)}=\frac{x+4}{x+5}.$$

EXAMPLE 3.—*Simplify* $\frac{x^2-4}{x^2+4x-12}$.

$$\frac{x^2-4}{x^2+4x-12}=\frac{(x+2)(x-2)}{(x+6)(x-2)}=\frac{x+2}{x+6}.$$

Examples VIIf

Simplify the following fractions :—

Lower.

1. $\frac{2x^2+13x+15}{x^2+6x+5}$.
2. $\frac{x(x+3)}{x^2+7x+12}$.
3. $\frac{x^2+x-20}{3x^2-10x-8}$.
4. $\frac{25x^2-16}{5x^2+11x-12}$.
5. $\frac{abx^2+acx}{b^2x^2+bcx}$.
6. $\frac{3x^2+3bx+7x+7b}{x+b}$.

Middle.

7. $\frac{2x^2+14x+24}{x+4}$.
8. $\frac{x^3+3x^2+2x}{x(x+2)}$.

Higher.

9. $\frac{x^2+ax+bx+ab}{2x^2+2ax+bx+ab}$.

108. EXAMPLE 1.—*Solve the equation* $x^2+8x+12=0$.

This equation cannot be solved by the method you have learnt for equations which have no term with x^2.

Suppose that we factorise the left-hand side of this equation.

We should then have $(x+6)(x+2)=0$.

Now, if two numbers multiplied together give the result zero, we know that one or other of the original numbers is zero, or both may be zero.

Therefore since $(x+6)$ multiplied by $(x+2)=0$,
either $x+2=0$ or $x+6=0$; or $x+2=0$, and $x+6=0$.

Thus we have formed two simple equations, from which we obtain the result $x=-6$ or $x=-2$.

109. We see that, when we have an equation of the type given in Example 1 of the previous paragraph, two different values of x both satisfy the required condition.

EXAMPLE 1.—*Solve* $6x^2-13x-28=0$.

$$6x^2-13x-28=0,$$

i.e.
$$6x^2-21x+8x-28=0$$
$$3x(2x-7)+4(2x-7)=0$$
$$(3x+4)(2x-7)=0;$$

$\therefore$ either $3x+4=0$ or $2x-7=0$; or $3x+4=0$ and $2x-7=0$.

If
$$3x+4=0$$
$$3x=-4,$$
$$\therefore\ x=-\tfrac{4}{3}.$$

If
$$2x-7=0$$
$$2x=7$$
$$x=\tfrac{7}{2},$$

$\therefore$ the solution of $6x^2-13x-28=0$ is $x=-\frac{4}{3}$ or $\frac{7}{2}$.

Examples VIIg

Solve the following equations :—

Lower.

1. $3x^2-20x-7=0$.
2. $x^2+x-20=0$.
3. $x^2-13x+42=0$.
4. $8x^2+18x+7=0$.
5. $8x^2-10x+3=0$.
6. $5x^2-33x+40=0$.
7. $54x^2-3x-2=0$.
8. $12x^2+25x+12=0$.
9. $77x^2+5x-28=0$.
10. $x^2+18x+72=0$.
11. $x^2+5x=0$.
12. $6x^2-12x=0$.

Middle.

13. $8x^2-20x+8=0$.
14. $3x^2-7x=0$.
15. $x^2-49=0$.
16. $x^2+2x+1=0$.
17. $x^2+8x+16=0$.
18. $9x^2+12x+4=0$.

Higher.

19. $12x^2-43x+36=0$.
20. $5x^2+22x+8=0$.
21. $6x^3-34x^2-56x=0$.

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